لجنة الميكانيك
تقدم لكم..

[المكتبة التخصصية]

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Chapter 2

2.1 $u_1^{(1)} = U_1 \quad u_1^{(2)} = u_2^{(1)} = U_2 \quad u_2^{(2)} = u_1^{(3)} = U_3 \quad u_2^{(3)} = U_4$

U_i = GLOBAL DISPLACEMENT i, $i = 1, 4$

$$[K] = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2+K_3 & -K_3 \\ 0 & 0 & -K_3 & K_3 \end{bmatrix}$$

2.2

$$[K] = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2+2K_3 & -2K_3 \\ 0 & 0 & -2K_3 & 2K_3 \end{bmatrix}$$

2.3

$$[K] = \begin{bmatrix} K_1 & -K_1 & 0 & 0 & \cdots & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 & \cdots & 0 \\ 0 & -K_2 & K_1+K_3 & -K_3 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & K_{N-2}+K_{N-1} & -K_{N-1} \\ 0 & 0 & \cdots & & -K_{N-1} & K_{N-1} \end{bmatrix}$$

2.4

$$[K] = \begin{bmatrix} 50 & -50 & 0 \\ -50 & 75 & -25 \\ 0 & -25 & 25 \end{bmatrix} \text{ LB/IN.}$$

$U_2 = \delta = 0.75$ IN.

$U_1 = 0 \longrightarrow$ DISPLACEMENT CONSTRAINT

$$\begin{bmatrix} 75 & -25 \\ -25 & 25 \end{bmatrix} \begin{Bmatrix} 0.75 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_3 \end{Bmatrix}$$

FIRST EQUATION: $75(0.75) - 25U_3 = 0 \Rightarrow U_3 = 2.25$ IN.

SECOND EQUATION: $-25(0.75) + 25U_3 = F_3 \Rightarrow F_3 = 37.5$ LB.

2.5

$$[K] = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2+K_3 & -K_3 \\ 0 & 0 & -K_3 & K_3 \end{bmatrix} = \begin{bmatrix} 30 & -30 & 0 & 0 \\ -30 & 70 & -40 & 0 \\ 0 & -40 & 70 & -30 \\ 0 & 0 & -30 & 30 \end{bmatrix} \text{LB/IN.}$$

$U_2 = 0 \quad U_4 = 1$ IN. $\quad U_1 = 0$ (BOUNDARY CONDITION)

$$\begin{bmatrix} 70 & -40 & 0 \\ -40 & 70 & -30 \\ 0 & -30 & 30 \end{bmatrix} \begin{Bmatrix} 0 \\ U_3 \\ 1 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ 0 \\ F_4 \end{Bmatrix}$$

$$\left.\begin{aligned} -40U_3 &= F_2 \\ 70U_3 - 30 &= 0 \\ -30U_3 + 30 &= F_4 \end{aligned}\right\} \quad \begin{aligned} U_3 &= 3/7 \text{ IN.} \\ F_2 &= 17.14 \text{ LB.} \\ F_4 &= 17.14 \text{ LB.} \end{aligned}$$

2.6 (a)

$$[K] = \begin{bmatrix} K & -K & 0 & 0 \\ -K & K+3K & -3K & 0 \\ 0 & -3K & 3K+2K & -2K \\ 0 & 0 & -2K & 2K \end{bmatrix} = \begin{bmatrix} K & -K & 0 & 0 \\ -K & 4K & -3K & 0 \\ 0 & -3K & 5K & -2K \\ 0 & 0 & -2K & 2K \end{bmatrix}$$

(b) $U_e = \frac{1}{2}k(U_2-U_1)^2 + \frac{1}{2}(3k)(U_3-U_2)^2 + \frac{1}{2}(2k)(U_4-U_3)^2$

$$\frac{\partial U_e}{\partial U_1} = F_1 = -k(U_2-U_1)$$

$$\frac{\partial U_e}{\partial U_2} = k(U_2-U_1) - (3k)(U_3-U_2) = F_2$$

$$\frac{\partial U_e}{\partial U_3} = (3k)(U_3-U_2) - (2k)(U_4-U_3) = F_3$$

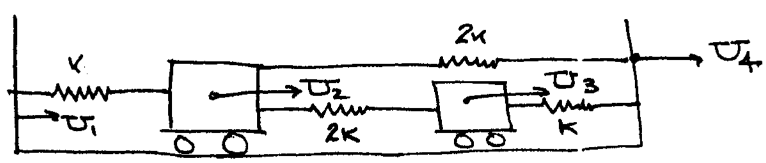
$$\frac{\partial U_e}{\partial U_4} = (2k)(U_4-U_3)$$

IN MATRIX FORM:

$$\begin{bmatrix} k & -k & 0 & 0 \\ -k & 4k & -3k & 0 \\ 0 & -3k & 5k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

RESULTS ARE IDENTICAL AS EXPECTED.

2.7



$$[K] = \begin{bmatrix} k & -k & 0 & 0 \\ -k & 5k & -2k & -2k \\ 0 & -2k & 3k & -k \\ 0 & -2k & -k & 3k \end{bmatrix} \qquad [K]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ -F_1 \\ F_2 \\ R_4 \end{Bmatrix}$$

R_1 AND R_4 ARE REACTIONS

(b) APPLYING THE CONSTRAINTS $U_1 = U_4 = 0$ AND SUBSTITUTING NUMERICAL VALUES

$$\begin{bmatrix} 250 & -100 \\ -100 & 150 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 15 \end{Bmatrix}$$

GIVES

$$U_2 = -0.055 \text{ IN.} \qquad U_3 = 0.064 \text{ IN.}$$

2.8 $$U_e = \tfrac{1}{2} k_1 (U_2 - U_1)^2 + \tfrac{1}{2} k_2 (U_3 - U_2)^2 + \tfrac{1}{2} k_3 (U_4 - U_3)^2 + \tfrac{1}{2} k_4 (U_5 - U_4)^2$$

$$\frac{\partial U_e}{\partial U_1} = -k_1 (U_2 - U_1) = F_1$$

$$\frac{\partial U_e}{\partial U_2} = k_1 (U_2 - U_1) - k_2 (U_3 - U_2) = F_2$$

$$\frac{\partial U_e}{\partial U_3} = k_2 (U_3 - U_2) - k_3 (U_4 - U_3) = F_3$$

$$\frac{\partial U_e}{\partial U_4} = k_3 (U_4 - U_3) - k_4 (U_5 - U_4) = F_4$$

$$\frac{\partial U_e}{\partial U_5} = -k_4 (U_5 - U_4) = F_1$$

2.9 $K_1 = K_2 = K_3 = K_4 = 10 \text{ N/mm}$ $\quad F_2 = 20\text{N}$ $\quad F_3 = 25\text{N}$

$F_4 = 40\text{ N}$ $\quad U_1 = U_5 = 0$

$$\begin{bmatrix} 10 & -10 & 0 & 0 & 0 \\ -10 & 20 & -10 & 0 & 0 \\ 0 & -10 & 20 & -10 & 0 \\ 0 & 0 & -10 & 20 & -10 \\ 0 & 0 & 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ -20 \\ 25 \\ 40 \\ R_5 \end{Bmatrix}$$

$$\begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 25 \\ 40 \end{Bmatrix} \Rightarrow \begin{matrix} U_2 = 0.75 \text{ mm} \\ U_3 = 3.5 \text{ mm} \\ U_4 = 3.75 \text{ mm} \end{matrix}$$

$R_1 = -10U_2 = -7.5 \text{ N}$

$R_5 = -10U_4 = 37.5 \text{ N}$

$f^{(1)} = R_1 = 7.5\text{N}$

$f^{(2)} = 10\,(U_3 - U_2) = 27.5 \text{ N}$

$f^{(3)} = 10\,(U_4 - U_3) = 2.5 \text{ N}$

$f^{(4)} = R_4 = 37.5 \text{ N}$

2.10 $K_e = \dfrac{AE}{L_e} = \dfrac{500(207)(10^3)}{500} = 207(10^3) \text{ N/mm}$

$$[K^{(1)}] = 207(10^3) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [K^{(2)}]$$

$$[K] = 207(10^3)\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$207(10^3)\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 12(10^3) \end{Bmatrix}$$

$$U_3 = \frac{2(12)}{207} = 0.116 \text{ MM} \qquad U_2 = \frac{12}{207} = 0.058 \text{ MM}$$

$$\sigma^{(1)} = 207(10)^3 \frac{0.058 - 0}{500} = 24 \text{ MPa}$$

$$\sigma^{(2)} = 207(10^3) \frac{0.116 - 0.058}{500} = 24 \text{ MPa}$$

BUCKLING SHOULD BE CONSIDERED.

2.11 $$[K^{(1)}] = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 3(10^6)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ LB/IN}$$

$$[K^{(2)}] = \frac{A_2 E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1.125(10^6)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ LB/IN}$$

$$[K] = 10^6\begin{bmatrix} 3 & -3 & 0 \\ -3 & 4.125 & -1.125 \\ 0 & -1.125 & 1.125 \end{bmatrix}$$

$$10^6\begin{bmatrix} 3 & -3 & 0 \\ -3 & 4.125 & -1.125 \\ 0 & -1.125 & 1.125 \end{bmatrix}\begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 20(10^3) \end{Bmatrix}$$

$$U_2 = \frac{20(10^3)}{3(10^6)} \cong 6.7(10^{-3}) \text{ IN.}$$

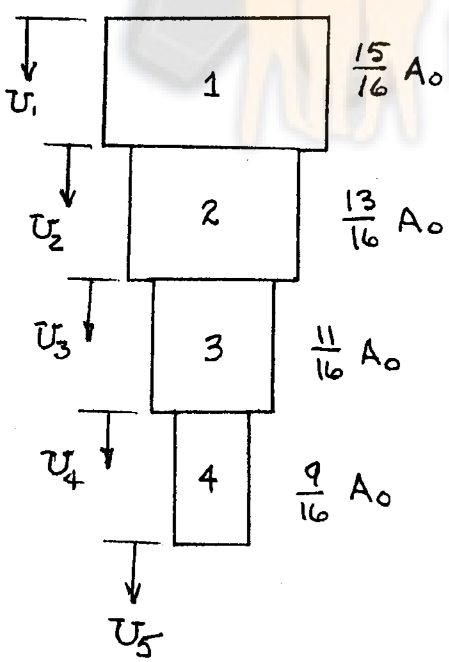
$$U_3 = \frac{1}{1.125(10^6)}\left(20(10^3) + 1.125(10^6)U_2\right) \cong 24.5(10^3) \text{ IN.}$$

$$R_1 = -3(10^6)U_2 = -20(10^3) \text{ LB.}$$

$$\sigma^{(1)} = 15(10^6)\frac{U_2 - U_1}{20} \cong 5025 \text{ PSI} \quad \text{(TENSILE)}$$

$$\sigma^{(2)} = 10(10^6)\frac{U_3 - U_2}{20} \cong 8900 \text{ PSI} \quad \text{(TENSILE)}$$

2.12



$E = 10(10^6) \text{ LB/IN}^2 \quad A_0 = 4 \text{ IN}^2$

$L_e = 5 \text{ IN} \quad P = 4000 \text{ LB.}$

$$\frac{A_0 E}{L_e} = \frac{4(10)(10^6)}{5} = 8(10^6) \text{ LB/IN}$$

$$[K^{(1)}] = \frac{8(10^6)}{16}\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$[K^{(2)}] = \frac{8(10^6)}{16}\begin{bmatrix} 13 & -13 \\ -13 & 13 \end{bmatrix}$$

$$[K^{(3)}] = \frac{8(10^6)}{16}\begin{bmatrix} 11 & -11 \\ -11 & 11 \end{bmatrix} \quad [K^{(4)}] = \frac{8(10^6)}{16}\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

ASSEMBLING THE SYSTEM EQUATIONS GIVES

$$\frac{8(10^6)}{16}\begin{bmatrix} 15 & -15 & 0 & 0 & 0 \\ -15 & 28 & -13 & 0 & 0 \\ 0 & -13 & 24 & -11 & 0 \\ 0 & 0 & -11 & 20 & -9 \\ 0 & 0 & 0 & -9 & 9 \end{bmatrix}\begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ 0 \\ 4000 \end{Bmatrix}$$

ELIMINATING THE REACTION EQUATION AND SOLVING THE REMAINING 4 x 4 SYSTEM GIVES

$U_2 = 5.33(10^{-4})$ IN. $\quad U_3 = 1.149(10^{-3})$ IN.

$U_4 = 1.876(10^{-3})$ IN. $\quad U_5 = 2.765(10^{-3})$ IN.

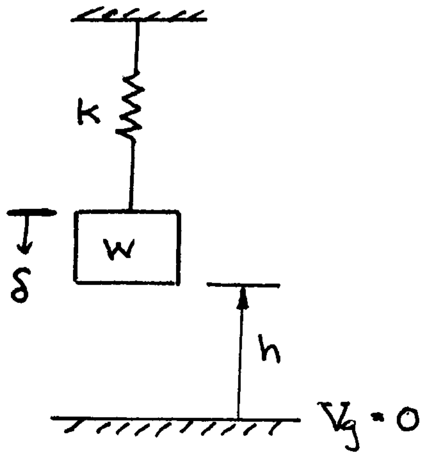
$$\sigma^{(1)} = 10(10^6)\frac{U_2 - U_1}{5} = 1066 \text{ psi}$$

$$\sigma^{(2)} = 10(10^6)\frac{U_3 - U_2}{5} = 1232 \text{ psi}$$

$$\sigma^{(3)} = 10(10^6)\frac{U_4 - U_3}{5} = 1454 \text{ psi}$$

$$\sigma^{(4)} = 10(10^6)\frac{U_5 - U_4}{5} = 1778 \text{ psi}$$

2.13



WHEN WEIGHT IS AT HEIGHT h ABOVE AN ARBITRARY DATUM FOR GRAVITATIONAL POTENTIAL ENERGY, THE SPRING IS UNDEFORMED. FOR ANY OTHER POSITION δ, POTENTIAL ENERGY IS

$$V = \tfrac{1}{2} k \delta^2 + W(h-\delta)$$

FOR MINIMUM POTENTIAL ENERGY

$$\frac{dV}{d\delta} = 0 = k\delta - W$$

$\therefore \quad k\delta = W$ AND THIS IS THE EQUILIBRIUM FORCE EQUATION.

2.14

CHECK THE REQUIRED NODAL CONDITIONS:

$$\left.\begin{aligned} N_1(0) &= \cos\frac{\pi 0}{2L} = 1 \\ N_1(L) &= \cos\frac{\pi}{2} = 0 \\ N_2(0) &= \sin\frac{\pi 0}{2L} = 0 \\ N_2(1) &= \sin\frac{\pi}{2} = 1 \end{aligned}\right\} \quad \text{AS REQUIRED}$$

$$\epsilon = \frac{du}{dx} = \frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 = \frac{\pi}{2L}\left(-u_1 \sin\frac{\pi x}{2L} + u_2 \cos\frac{\pi x}{2L}\right)$$

$$\epsilon(x=0) = \frac{\pi}{2L}u_2 \qquad \epsilon(x=L) = -\frac{\pi}{2L}u_1$$

THE STRAIN VARIATION IS CLEARLY NOT ACCEPTABLE

PHYSICALLY. THIS IS A CASE IN WHICH A SET OF "ADMISSABLE FUNCTIONS" SATISFY THE BOUNDARY CONDITIONS BUT ARE NOT IN ACCORD WITH THE PHYSICS OF THE PROBLEM (CHAPTER 5).

2.15 FROM MECHANICS OF MATERIALS WE HAVE

$$\theta = \theta_2 - \theta_1 = \frac{TL}{JG} \quad (1)$$

AND

$$\tau = \frac{Tr}{J} \quad (2)$$

THE SHEAR STRAIN IS $\gamma = \frac{\tau}{G} = \frac{Tr}{JG} \quad (3)$

TOTAL STRAIN ENERGY IS

$$U_e = \frac{1}{2}\iiint_V \tau\gamma \, dV = \frac{1}{2}\iiint_V \frac{T^2 r^2}{J^2 G} \, dV$$

USING (1) TO EXPRESS TORQUE IN TERMS OF ANGULAR DISPLACEMENTS WE OBTAIN

$$U_e = \frac{G}{2L^2}\int_0^L (\theta_2 - \theta_1)^2 \iint_A r^2 \, dA \, dx$$

RECOGNIZING

$$\iint_A r^2 dA = J = \text{POLAR MOMENT OF INERTIA}$$

$$U_e = \frac{JG}{2L^2}\int_0^L (\theta_2-\theta_1)^2 dx = \frac{JG}{2L}(\theta_2-\theta_1)^2$$

THEN

$$\Pi_p = U_e - W = \frac{JG}{2L}(\theta_2-\theta_1)^2 - T_1\theta_1 - T_2\theta_2$$

$$\frac{\partial \Pi_p}{\partial \theta_1} = 0 = \frac{JG}{L}(\theta_2-\theta_1)(-1) - T_1$$

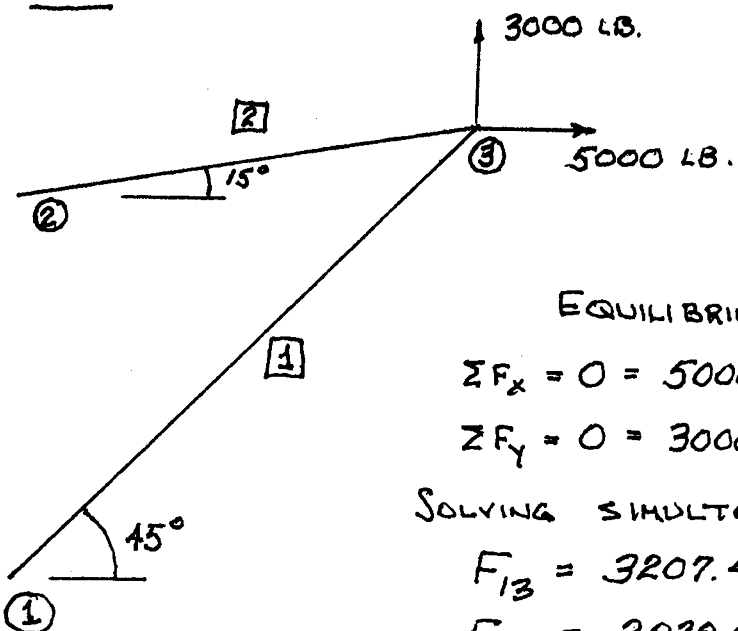
$$\frac{\partial \Pi_p}{\partial \theta_2} = 0 = \frac{JG}{L}(\theta_2-\theta_1) - T_2$$

IN MATRIX FORM

$$\frac{JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Chapter 3

3.1



Equilibrium at node 3:

$$\Sigma F_x = 0 = 5000 - F_{13} \cos 45 - F_{23} \cos 15$$

$$\Sigma F_y = 0 = 3000 - F_{13} \sin 45 - F_{23} \sin 15$$

Solving simultaneously yields:

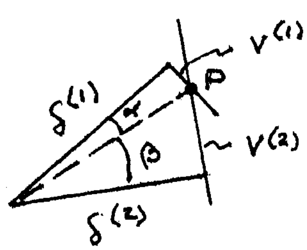
$$F_{13} = 3207.4 \text{ LB}$$

$$F_{23} = 2828.4 \text{ LB}$$

$$\delta^{(1)} = \frac{F_{13}}{K} = \frac{3207.4}{52000} = 0.0617 \text{ IN.}$$

$$\delta^{(2)} = \frac{F_{23}}{K} = \frac{2828.4}{52060} = 0.0544 \text{ IN.}$$

In order to maintain displacement continuity at node 3, each element must rotate



P is the actual equilibrium position

$$\frac{\delta^{(1)}}{\delta^{(2)}} = \frac{\cos\alpha}{\cos\beta} \qquad \alpha + \beta = 30^\circ$$

$$\frac{\cos\alpha}{\cos(30-\alpha)} = \frac{0.0617}{0.0544} = 1.13419$$

$$\alpha = 1.8^\circ \qquad \beta = 28.2^\circ$$

$$v^{(1)} = \delta^{(1)} \tan\alpha \cong 0.0019 \text{ IN.}$$

$$v^{(2)} = \delta^{(2)} \tan\beta = 0.0292 \text{ IN.}$$

THE ACTUAL DISPLACEMENT COMPONENTS OF NODE 3 ARE THEN

$$\delta_x = \delta^{(2)} \cos 15 - v^{(2)} \sin 15 = 0.0450 \text{ IN.}$$

$$\delta_y = \delta^{(2)} \sin 15 + v^{(2)} \cos 15 = 0.0423 \text{ IN.}$$

<u>3.2</u> IN THE ELEMENT COORDINATE SYSTEMS:

$$[k^{(1)}] = [k^{(2)}] = 5.2(10^4)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

FOR ELEMENT 1, $\theta = 45^\circ$ SO PER EQ. 3.28

$$[K^{(1)}] = 2.6(10^4) \begin{array}{c} \begin{array}{cccc} ① & ② & ⑤ & ⑥ \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array} \text{ LB/IN}$$

FOR ELEMENT 2 USING $\theta = 15°$

$$[K^{(2)}] = 5.2(10^4)\begin{array}{c} \begin{array}{cccc} ③ & ④ & ⑤ & ⑥ \end{array} \\ \begin{bmatrix} 0.933 & 0.25 & -0.933 & -0.25 \\ 0.25 & 0.067 & -0.25 & -0.067 \\ -0.933 & -0.25 & 0.933 & 0.25 \\ -0.25 & -0.067 & 0.25 & 0.067 \end{bmatrix} \end{array}$$

ASSEMBLING ONLY THE PORTION OF THE GLOBAL STIFFNESS MATRIX ASSOCIATED WITH U_5 AND U_6 WE OBTAIN

$$2.6(10^4)\begin{bmatrix} 2.866 & 1.5 \\ 1.5 & 1.134 \end{bmatrix}\begin{Bmatrix} U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 5000 \\ 3000 \end{Bmatrix}$$

FROM WHICH $U_5 = 0.0450$ IN.

$U_6 = 0.0423$ IN.

$$\begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix} = [R^{(1)}]\begin{Bmatrix} U_1 \\ U_2 \\ U_5 \\ U_6 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.0450 \\ 0.0423 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.0617 \end{Bmatrix} \text{ IN.}$$

$f^{(1)} = 52000(0.0617) = 3209$ LB.

RESULTS FOR ELEMENT 2 ARE ALSO IDENTICAL TO PROBLEM 3.1 SOLUTION

3.3

$$[K^{(e)}] = k^{(e)} \begin{bmatrix} c\theta & 0 \\ s\theta & 0 \\ 0 & c\theta \\ 0 & s\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c\theta & s\theta & 0 & 0 \\ 0 & 0 & c\theta & s\theta \end{bmatrix}$$

$$= k^{(e)} \begin{bmatrix} c\theta & 0 \\ s\theta & 0 \\ 0 & c\theta \\ 0 & s\theta \end{bmatrix} \begin{bmatrix} c\theta & s\theta & -c\theta & -s\theta \\ -c\theta & -s\theta & c\theta & s\theta \end{bmatrix}$$

$$= k^{(e)} \begin{bmatrix} c^2\theta & c\theta s\theta & -c^2\theta & -c\theta s\theta \\ c\theta s\theta & s^2\theta & -c\theta s\theta & -s^2\theta \\ -c^2\theta & -c\theta s\theta & c^2\theta & c\theta s\theta \\ -c\theta s\theta & -s^2\theta & c\theta s\theta & s^2\theta \end{bmatrix}$$

3.4

A MATRIX IS SINGULAR IF THE DETERMINANT IS ZERO. THE DETERMINANT CAN BE EXPRESSED IN TERMS OF THE MINORS (APPENDIX A). CALCULATION OF <u>ANY</u> MINOR IN EQ. 3.28 RESULTS IN ZERO, THUS THE MATRIX IS SINGULAR.

3.5 FOR EACH ELEMENT: $A = \frac{\pi}{4}(1.5)^2 = 1.767 \text{ in}^2$

$E = 30(10^6) \text{ lb/in}^2 \qquad L = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]^{1/2}$

$\theta = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$

THE ELEMENT STIFFNESS MATRICES ARE COMPUTED VIA EQ. 3.28.

(a) $L = 42.43$ IN. $\quad \theta = 45°$

$$\left[K^{(e)}\right] = 0.6247(10^6)\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

(b) $L = 11.180$ IN. $\quad \theta = 25.57°$

$$\left[K^{(e)}\right] = 10^6\begin{bmatrix} 3.7931 & 1.8965 & -3.7931 & -1.8965 \\ 1.8965 & 0.9483 & -1.8965 & -0.9483 \\ -3.7931 & -1.8965 & 3.7931 & 1.8965 \\ -1.8965 & -0.9483 & 1.8965 & 0.9483 \end{bmatrix}$$

(c) $L = 30.414$ IN. $\quad \theta = 80.54°$

$$\left[K^{(e)}\right] = 10^6\begin{bmatrix} 0.0471 & 0.2826 & -0.0471 & -0.2826 \\ & 1.6959 & -0.2826 & -1.6959 \\ & \text{SYM} & 0.0471 & 0.2826 \\ & & & 1.6959 \end{bmatrix}$$

(d) $L = 36.06$ in. $\theta = 146.31°$

$$[K^{(e)}] = 10^6 \begin{bmatrix} 1.0179 & -0.6786 & -1.0179 & 0.6786 \\ & 0.4524 & 0.6786 & -0.4524 \\ & \text{SYM} & 1.0179 & -0.6786 \\ & & & 0.4524 \end{bmatrix}$$

(e) $L = 41.231$ in. $\theta = -14.04°$

$$[K^{(e)}] = 10^6 \begin{bmatrix} 1.2101 & -0.3025 & -1.2101 & 0.3025 \\ & 0.0756 & 0.3025 & -0.0756 \\ & \text{SYM} & 1.2101 & -0.3025 \\ & & & 0.0756 \end{bmatrix}$$

3.6 $A = \frac{\pi}{4}(40)^2 = 1256.7\ mm^2$ $E = 69\,(10^3)$ MPa

(a)

$$[K^{(e)}] = \begin{bmatrix} 246.8 & 82.3 & -246.8 & -82.3 \\ & 27.4 & -82.3 & -27.4 \\ & \text{SYM} & 246.8 & 82.3 \\ & & & 27.4 \end{bmatrix} (10^3)\ \text{N/mm}$$

FOR (a) $L = 0.316$ m $\theta = 18.43°$

(b) $L = 0.283 \text{ m} \quad \theta = -45°$

$$[K^{(e)}] = \begin{bmatrix} 153.3 & -153.3 & -153.3 & 153.3 \\ & 153.3 & 153.3 & -153.3 \\ & \text{SYM} & 153.3 & -153.3 \\ & & & 153.3 \end{bmatrix} (10^3) \text{ N/mm}$$

(c) $L = 2.14 \text{ m} \quad \theta = 52.6°$

$$[K^{(e)}] = \begin{bmatrix} 14.95 & 19.55 & -14.95 & -19.55 \\ & 25.57 & -19.55 & -25.57 \\ & \text{SYM} & 14.95 & 19.55 \\ & & & 25.57 \end{bmatrix} (10^3) \text{ N/mm}$$

(d) $L = 1.92 \text{ m} \quad \theta = -141.3°$

$$[K^{(e)}] = \begin{bmatrix} 27.52 & 22.02 & -27.52 & -22.02 \\ & 17.62 & -22.02 & -17.62 \\ & \text{SYM} & 27.52 & 22.02 \\ & & & 17.62 \end{bmatrix} (10^3) \text{ N/mm}$$

(e) $L = 5 \text{ m} \quad \theta = 53.1°$

$$[K^{(e)}] = \begin{bmatrix} 6.24 & 8.32 & -6.24 & -8.32 \\ & 11.10 & -8.32 & -11.10 \\ & \text{SYM} & 6.24 & 8.32 \\ & & & 11.10 \end{bmatrix} (10^3) \text{ N/mm}$$

3.7a

ELEMENT

GLOBAL DISPLACEMENT

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1																			
2	2	2																			
3	3		1	1																	
4	4		2	2																	
5		3	3		1	1															
6		4	4		2	2															
7				3	3		1	1	1												
8				4	4		2	2	2												
9						3	3			1											
10						4	4			2											
11								3			1	1									
12								4			2	2									
13									3	3	3		1	1							
14									4	4	4		2	2							
15												3	3		1	1	1				
16												4	4		2	2	2				
17														3	3			1			
18														4	4			2			
19																3			1	1	
20																4			2	2	
21																	3	3	3		1
22																	4	4	4		2
23																				3	3
24																				4	4

3.7(b)

Global Displacement \ Element	1	2	3	4	5	6	7	8	9	10	11
1	1	1									
2	2	2									
3	3		1								
4	4		2								
5		3	3	1	1	1					1
6		4	4	2	2	2					2
7				3			1				3
8				4			2				4
9					3			1	1		
10					4			2	2		
11						3	3	3		1	
12						4	4	4		2	
13									3	3	
14									4	4	

3.7(c)

Global Displacement \ Element	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1											
2	2	2											
3		3	1	1	1								
4		4	2	2	2								
5					3	1							
6					4	2							
7	3		3				1		1				
8	4		4				2		2				
9							3	1		1	1		
10							4	2		2	2		
11				3		3		3				1	
12				4		4		4				2	
13									3	3			1
14									4	4			2
15											3	3	3
16											4	4	4

3.7(d)

ELEMENT

GLOBAL DISPLACEMENT	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	1															
2	2	2															
3	3		1	1													
4	4		2	2													
5		3	3		1	1											
6		4	4		2	2											
7				3	3		1	1									
8				4	4		2	2									
9						3	3		1								
10						4	4		2								
11									3	1							
12									4	2							
13								3			1		1	1			
14								4			2		2	2			
15										3	3	1					
16										4	4	2					
17												3	3		1		1
18												4	4		2		2
19														3	3	1	
20														4	4	2	
21																3	3
22																4	4

3.7(e)

ELEMENT

GLOBAL DISPLACEMENT	1	2	3	4	5	6	7
1	1		1	1			
2	2		2	2			
3	3	1					
4	4	2					
5		3	3		1	1	
6		4	4		2	2	
7				3	3		1
8				4	4		2
9						3	3
10						4	4

3.8

(a)

	1	2	3	4
L1	1	2	3	4
L2	1	2	5	6
L3	3	4	5	6
L4	3	4	7	8
L5	5	6	7	8
L6	5	6	9	10
L7	7	8	9	10
L8	7	8	11	12
L9	7	8	13	14
L10	9	10	13	14
L11	11	12	13	14
L12	11	12	15	16
L13	13	14	15	16
L14	13	14	17	18
L15	15	16	17	18
L16	15	16	19	20
L17	15	16	21	22
L18	17	18	21	22
L19	19	20	21	22
L20	19	20	23	24
L21	21	22	23	24

(b)

	1	2	3	4
L1	1	2	3	4
L2	1	2	5	6
L3	3	4	5	6
L4	3	4	7	8
L5	5	6	9	10
L6	5	6	11	12
L7	7	8	11	12
L8	9	10	11	12
L9	9	10	13	14
L10	11	12	13	14
L11	5	6	7	8

(c)

	1	2	3	4
L1	1	2	7	8
L2	1	2	3	4
L3	3	4	7	8
L4	3	4	11	12
L5	3	4	5	6
L6	5	6	11	12
L7	7	8	9	10
L8	9	10	11	12
L9	7	8	13	14
L10	9	10	13	14
L11	9	10	15	16
L12	11	12	15	16
L13	13	14	15	15

(d)

	1	2	3	4
L1	1	2	3	4
L2	1	2	5	6
L3	3	4	5	6
L4	3	4	7	8
L5	5	6	7	8
L6	5	6	9	10
L7	7	8	9	10
L8	7	8	13	14
L9	9	10	11	12
L10	11	12	15	16
L11	13	14	15	16
L12	15	16	17	18
L13	13	14	17	18
L14	13	14	19	20
L15	17	18	19	20
L16	19	20	21	22
L17	17	18	21	22

(e)

	1	2	3	4
L1	1	2	3	4
L2	3	4	5	6
L3	1	2	5	6
L4	1	2	7	8
L5	5	6	7	8
L6	5	6	9	10
L7	7	8	9	10

3.9 $\{u\} = [R]\{v\}$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} C\theta & S\theta & 0 & 0 \\ 0 & 0 & C\theta & S\theta \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

$$\epsilon = \frac{u_2 - u_1}{L} \qquad \sigma = E\epsilon$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \frac{AE}{L} = 1.875(10^5) \text{ LB/IN}$$

FOR (a), (b), (c) ONLY ANGLE θ DIFFERS.

(a) $\theta = 45°$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.099 \\ 0.233 \end{Bmatrix} \text{ IN.} \qquad \epsilon = 3.35 \times 10^{-3} \qquad \sigma = 33{,}500 \text{ psi}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = 1.875(10^5) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.099 \\ 0.233 \end{Bmatrix} = \begin{Bmatrix} -25125 \\ 25125 \end{Bmatrix} \text{ LB.}$$

(b) $\theta = 30°$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.053 \\ 0.101 \end{Bmatrix} \text{ IN.} \qquad \epsilon = 1.2 \times 10^{-3} \qquad \sigma = 12000 \text{ psi}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = 1.875(10^5) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.053 \\ 0.101 \end{Bmatrix} = \begin{Bmatrix} -9000 \\ 9000 \end{Bmatrix} \text{ LB.}$$

(c) $\theta = 110°$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.0017 \\ 0.0589 \end{Bmatrix} \text{ IN.} \qquad \epsilon = 1.43(10^{-3}) \qquad \sigma = 14{,}300 \text{ PSI}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = 1.875(10^5)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -10725 \\ 10725 \end{Bmatrix} \text{ LB.}$$

3.10 $L_1 = [30^2 + 10^2]^{1/2} = 31.62$

$L_2 = [10^2 + 10^2]^{1/2} = 14.14$

$\theta_1 = \tan^{-1}\left(\frac{-10}{30}\right) = -18.43°$ $\qquad \theta_2 = \tan^{-1}\left(\frac{-10}{10}\right) = -135°$

UTILIZING EQ. 3.28 THE ELEMENT STIFFNESS MATRICES IN THE GLOBAL FRAME ARE:

$$[K^{(1)}] = \begin{bmatrix} 0.4269 & -0.1423 & -0.4269 & 0.1423 \\ & 0.0474 & 0.1423 & -0.0474 \\ & & 0.4269 & -0.1423 \\ \text{SYM} & & & 0.0474 \end{bmatrix}(10^6)$$

$$[K^{(2)}] = \begin{bmatrix} 0.5303 & 0.5303 & -0.5303 & -0.5303 \\ & 0.5303 & -0.5303 & -0.5303 \\ & & 0.5303 & 0.5303 \\ \text{SYM} & & & 0.5303 \end{bmatrix}(10^6)$$

THE ASSEMBLED GLOBAL STIFFNESS MATRIX IS:

$$[K] = \begin{bmatrix} 0.4269 & -0.1423 & -0.4269 & 0.1423 & 0 & 0 \\ & 0.0474 & 0.1423 & -0.0474 & 0 & 0 \\ & & 0.9572 & 0.3880 & -0.5303 & -0.5303 \\ & & & 0.5777 & -0.5303 & -0.5303 \\ & \text{SYM} & & & 0.5303 & 0.5303 \\ & & & & & 0.5303 \end{bmatrix}$$

$$\times (10^6) \text{ LB/IN}$$

(b) APPLYING THE CONSTRAINT CONDITIONS $U_1 = U_2 = U_5 = U_6 = 0$ RESULTS IN

$$\begin{bmatrix} 0.9572 & 0.3880 \\ 0.3880 & 0.5777 \end{bmatrix} (10^6) \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1500 \end{Bmatrix}$$

SIMULTANEOUS SOLUTION GIVES THE NONCONSTRAINED DISPLACEMENTS AS

$$U_3 = 1.45 (10^{-3}) \text{ IN.} \qquad U_4 = -3.57 (10^{-3}) \text{ IN.}$$

3.11 $A = (15)^2 = 225 \text{ mm}^2$

$L^{(1)} = L^{(3)} = L^{(4)} = 1.5 \text{ m} \qquad L^{(2)} = 1.5\sqrt{2} \cong 2.12 \text{ m}$

$$k^{(1)} = k^{(3)} = k^{(4)} = \frac{225(69)(10^3)}{1500} = 10.35(10^3) \text{ N/mm}$$

$$k^{(2)} = \frac{225(69)(10^3)}{2120} = 7.32(10^3) \text{ N/mm}$$

THE ELEMENT STIFFNESS MATRICES ARE:

$$[k^{(1)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} 10.35(10^3)$$

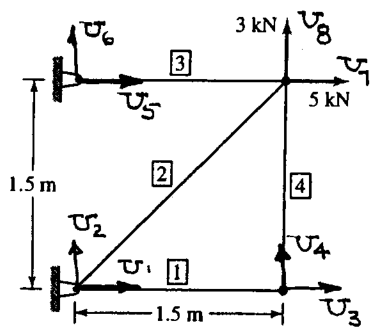
$$[k^{(2)}] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} 7.32(10^3)$$

TRANSFORMING ELEMENT MATRICES TO THE GLOBAL SYSTEM USING $\theta_1 = \theta_3 = 0$, $\theta_4 = 90°$, $\theta_2 = 45°$ RESULTS IN

$$[K^{(1)}] = [K^{(3)}] = \begin{bmatrix} 10.35 & 0 & -10.35 & 0 \\ 0 & 0 & 0 & 0 \\ -10.35 & 0 & 10.35 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (10^3)$$

$$[K^{(4)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10.35 & 0 & -10.35 \\ 0 & 0 & 0 & 0 \\ 0 & -10.35 & 0 & 10.35 \end{bmatrix} (10^3)$$

$$[K^{(2)}] = \begin{bmatrix} 3.66 & 3.66 & -3.66 & -3.66 \\ 3.66 & 3.66 & -3.66 & -3.66 \\ -3.66 & -3.66 & 3.66 & 3.66 \\ -3.66 & -3.66 & 3.66 & 3.66 \end{bmatrix} (10^3)$$



USING THE GLOBAL DISPLACEMENTS AS SHOWN, THE ELEMENT-TO-GLOBAL RELATIONS ARE

$[L_1] = [1 \;\; 2 \;\; 3 \;\; 4]$

$[L_2] = [1 \;\; 2 \;\; 7 \;\; 8]$

$[L_3] = [5 \;\; 6 \;\; 7 \;\; 8]$

$[L_4] = [3 \;\; 4 \;\; 7 \;\; 8]$

APPLICATION OF THE DIRECT ASSEMBLY PROCEDURE GIVES THE 8×8 GLOBAL STIFFNESS MATRIX:

$$[K] = \begin{bmatrix} 14.01 & 3.66 & -10.35 & 0 & 0 & 0 & -3.66 & 3.66 \\ & 3.66 & 0 & 0 & 0 & 0 & -3.66 & -3.66 \\ & & 10.35 & 0 & 0 & 0 & 0 & 0 \\ & & & 10.35 & 0 & 0 & 0 & -10.35 \\ & & & & 10.35 & 0 & -10.35 & 0 \\ & \text{SYM} & & & & 0 & 0 & 0 \\ & & & & & & 14.01 & 3.66 \\ & & & & & & & 14.01 \end{bmatrix}$$

$$\times (10^3) \ \text{N/mm}$$

THE CONSTRAINT CONDITIONS ARE $U_1 = U_2 = U_5 = U_6 = 0$

SO THE EQUATIONS FOR "ACTIVE" DISPLACEMENTS ARE

$$10^3 \begin{bmatrix} 10.35 & 0 & 0 & 0 \\ & 10.35 & 0 & -10.35 \\ \text{SYM} & & 14.01 & 3.66 \\ & & & 14.01 \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{Bmatrix} (10^3)$$

SIMULTANEOUS SOLUTION GIVES THE DISPLACEMENTS

$U_3 = 0$ (OBVIOUS VIA EXAMINATION OF LOADING)

$U_4 = 0.626$ mm

$U_7 = 0.193$ mm

$U_8 = 0.626$ mm

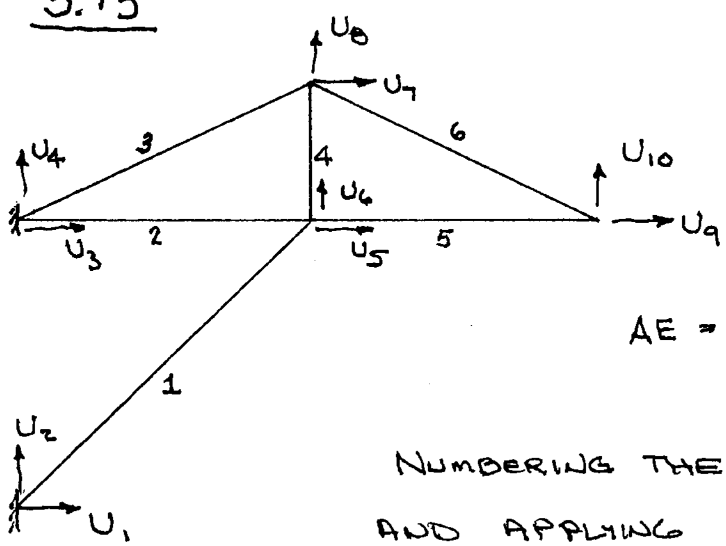
3.12 REMOVING ELEMENTS 1 AND 4 GIVES

$$10^3\begin{bmatrix}14.01 & 3.66\\ 3.66 & 3.66\end{bmatrix}\begin{Bmatrix}U_7\\ U_8\end{Bmatrix} = \begin{Bmatrix}5\\ 3\end{Bmatrix}10^3$$

$U_7 = 0.193$ mm $\quad U_8 = 0.626$ mm

(ELEMENTS 1 AND 4 ARE REDUNDANT.)

3.13



$AE = .75(3.5)(2)10^6 = 12.25(10^6)$

NUMBERING THE ELEMENTS AS SHOWN AND APPLYING EQ. 3.28 TO EACH ELEMENT:

$L^{(1)} = 96\sqrt{2}$ IN. $\quad \theta_1 = 45^\circ$

$$[K^{(1)}] = 10^4\begin{bmatrix}4.51 & 4.51 & -4.51 & -4.51\\ & 4.51 & -4.51 & -4.51\\ \text{SYM} & & 4.51 & 4.51\\ & & & 4.51\end{bmatrix} \text{ LB/IN}$$

$L^{(2)} = L^{(5)} = 96 \text{ in.} \qquad \theta_2 = \theta_5 = 0$

$$[K^{(2)}] = [K^{(5)}] = 10^4 \begin{bmatrix} 12.76 & 0 & -12.76 & 0 \\ 0 & 0 & 0 & 0 \\ -12.76 & 0 & 12.76 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ lb/in}$$

$L^{(3)} = 110.85 \text{ in} \qquad \theta_3 = 30^\circ$

$$[K^{(3)}] = 10^4 \begin{bmatrix} 8.29 & 4.78 & -8.29 & -4.78 \\ & 2.76 & -4.78 & -2.76 \\ \text{SYM} & & 8.29 & 4.78 \\ & & & 2.76 \end{bmatrix} \text{ lb/in}$$

$L^{(4)} = 55.4 \text{ in.} \qquad \theta_4 = 90^\circ$

$$[K^{(4)}] = 10^4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 22.11 & 0 & -22.11 \\ \text{SYM} & & 0 & 0 \\ & & & 22.11 \end{bmatrix} \text{ lb/in}$$

$L^{(6)} = 110.85 \text{ in.} \qquad \theta_6 = -30^\circ$

$$[K^{(6)}] = 10^4 \begin{bmatrix} 8.29 & -4.78 & -8.29 & 4.78 \\ & 2.76 & 4.78 & -2.76 \\ & & 8.29 & -4.78 \\ & & & 2.76 \end{bmatrix} \text{ lb/in}$$

THE ELEMENT TO GLOBAL DISPLACEMENT RELATIONS ARE:

$$[L_1] = [1 \quad 2 \quad 5 \quad 6]$$
$$[L_2] = [3 \quad 4 \quad 5 \quad 6]$$
$$[L_3] = [3 \quad 4 \quad 7 \quad 8]$$
$$[L_4] = [5 \quad 6 \quad 7 \quad 8]$$
$$[L_5] = [5 \quad 6 \quad 9 \quad 10]$$
$$[L_6] = [7 \quad 8 \quad 9 \quad 10]$$

RESULTING IN THE GLOBAL STIFFNESS MATRIX

	1	2	3	4	5	6	7	8	9	10
1	4.51	4.51	0	0	-4.51	-4.51	0	0	0	0
2		4.51	0	0	-4.51	-4.51	0	0	0	0
3			21.05	4.78	-12.76	0	-8.29	-4.78	0	0
4				2.76	0	0	-4.78	-2.76	0	0
5					30.03	4.51	0	0	-12.76	0
6						26.52	0	-22.11	0	0
7		SYM					16.58	0	-8.29	4.78
8								27.63	4.78	-2.76
9									21.05	-4.78
10										2.76

(10^4) LB/IN

APPLYING THE CONSTRAINTS $U_1 = U_2 = U_3 = U_4 = 0$ AND APPLIED LOAD $F_{10} = -500$ LB. THE REDUCED SYSTEM IS:

$$10^4 \begin{bmatrix} 30.03 & 4.51 & 0 & 0 & -12.76 & 0 \\ & 26.62 & 0 & -22.11 & 0 & 0 \\ & & 16.58 & 0 & -8.29 & 4.78 \\ & & & 27.63 & 4.78 & -2.76 \\ & \text{SYM} & & & 21.05 & -4.78 \\ & & & & & 2.76 \end{bmatrix} \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \\ U_{10} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -500 \end{Bmatrix}$$

YIELDING

$$U_5 = 0.0010 \text{ in.}$$
$$U_6 = -0.0231 \text{ in.}$$
$$U_7 = 0.0263 \text{ in.}$$
$$U_8 = -0.0276 \text{ in.}$$
$$U_9 = -0.0057 \text{ in.}$$
$$U_{10} = -0.1013 \text{ in.}$$

THE REACTION FORCES ARE

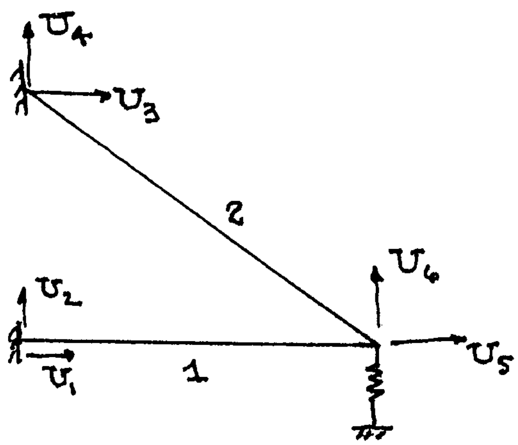
$$R_1 = R_2 = -4.51(10^4)(U_5 + U_6) = 996.7 \text{ lb.}$$
$$R_3 = -(12.76 U_5 + 8.29 U_7 + 4.78 U_8)10^4 = -988.6 \text{ lb.}$$
$$R_4 = -(4.78 U_7 + 2.76 U_8)10^4 = -495.4 \text{ lb.}$$

EQUILIBRIUM:

$$\Sigma F_x = R_1 + R_3 = -1.9 \text{ lb.}$$
$$\Sigma F_y = R_2 + R_4 - 500 = 1.3 \text{ lb.}$$

SO EQUILIBRIUM IS REASONABLE GIVEN THE ACCURACY OF MANUAL SOLUTION.

3.14



$A = \frac{\pi}{4}(20)^2 = 314.16 \ mm^2$

$E = 80(10^3) \ MPa$

$L^{(1)} = 4000 \ mm \quad \theta_1 = 0°$

$\frac{AE}{L^{(1)}} = 6.28(10^3) \ N/mm$

$$[K^{(1)}] = \begin{bmatrix} 6.28 & 0 & -6.28 & 0 \\ 0 & 0 & 0 & 0 \\ -6.28 & 0 & 6.28 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} 10^3$$

$L^{(2)} = 5000 \ mm \quad \theta_2 = -36.87° \quad \frac{AE}{L^{(2)}} = 5.03(10^3) \ N/mm$

$$[K^{(2)}] = \begin{bmatrix} 3.22 & -2.41 & -3.22 & 2.41 \\ -2.41 & 1.81 & 2.41 & -1.81 \\ -3.22 & 2.41 & 3.22 & -2.41 \\ 2.41 & -1.81 & -2.41 & 1.81 \end{bmatrix} 10^3$$

ASSEMBLING THE GLOBAL STIFFNESS MATRIX (WHILE NOTING THAT THE SPRING STIFFNESS SIMPLY ADDS TO K_{66}) GIVES:

$$\begin{bmatrix} 6.28 & 0 & 0 & 0 & -6.28 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 3.22 & -2.41 & -3.22 & 2.41 \\ & & & 1.81 & 2.42 & -1.81 \\ & \text{SYM} & & & 9.50 & -2.41 \\ & & & & & 1.86 \end{bmatrix} 10^3$$

APPLYING THE CONSTRAINTS $U_1 = U_2 = U_3 = U_4 = 0$

AND NODAL FORCES

$$F_5 = 15(10^3)\cos 50 = 9.64(10^3)\ N$$

$$F_6 = 15(10^3)\sin 50 = 11.49(10^3)\ N$$

WE HAVE

$$\begin{bmatrix} 9.50 & -2.41 \\ -2.41 & 1.86 \end{bmatrix} \begin{Bmatrix} U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 9.64 \\ 11.49 \end{Bmatrix}$$

AND

$$U_5 = 3.86\ \text{mm}$$

$$U_6 = 11.18\ \text{mm}$$

THE REACTION FORCES ARE

$$R_1 = -6.28(10^3)U_5 = -24241\ N$$

$$R_2 = 0$$

$$R_3 = (-3.22U_5 + 2.41U_6)10^3 = 14515\ N$$

$$R_4 = (2.42U_5 - 1.81U_6)10^3 = -10895\ N$$

AND THE SPRING FORCE IS

$$F_s = -50U_6 = -559\ N \quad (\text{Y-DIRECTION})$$

EQUILIBRIUM:

$$\Sigma F_X = R_1 + R_3 + 9640 = -.86 \text{ N}$$

$$\Sigma F_Y = R_2 + R_4 + F_5 + 1149.0 = .36 \text{ N}$$

EQUILIBRIUM ERROR IS LESS THAN 1% OF APPLIED LOAD.

THE EXTERNAL WORK IS

$$W = \frac{1}{2}(F_5 U_5 + F_6 U_6) = 82.9(10^3) \text{ N-mm}$$

ELEMENT STRAIN ENERGY IS CALCULATED VIA

$$U_e^{(e)} = \frac{1}{2}\{U\}^T[K^{(e)}]\{U\}$$

FOR ELEMENT 1, THIS IS SIMPLY

$$U_e^{(1)} = \frac{1}{2} K^{(1)} U_5^2 = \frac{1}{2}(6.28)10^3(3.86)^2$$

$$\cong 46.8\,(10^3) \text{ N-mm}$$

FOR ELEMENT 2:

$$U_e^{(2)} = \frac{1}{2}\left[3.22\,(3.86)^2 + 2(-2.41)(3.86)(11.18) + 1.81(11.18)^2\right]$$

$$= 33.1\,(10^3) \text{ N-mm}$$

FOR THE SPRING $\quad U_e^{(3)} = \frac{1}{2}(50)(11.18)^2 = 3.1(10^3)$ N-mm

TOTAL:

$$U_e = 83\,(10^3) \text{ N-mm}$$

THEREFORE BALANCE IS REASONABLE.

3.15 REMOVING THE SPRING RESULTS IN THE DISPLACEMENTS.

$$U_5 = 3.98 \text{ mm}$$
$$U_6 = 11.65 \text{ mm}$$

HENCE THE SPRING EFFECTS ARE MINOR.

3.16 IN THIS INSTANCE, INSTEAD OF THE CONSTRAINT CONDITION $U_1 = 0$, WE HAVE AN IMPOSED DISPLACEMENT $U_1 = -0.5''$

REFERRING TO THE GLOBAL STIFFNESS MATRIX FOR PROBLEM 3.13, THE MATHEMATICAL EFFECT IS TO ADD NODAL FORCES

$$F_5 = F_6 = -4.51(10^4)(0.5) \cong -2.26(10^4)$$

ADDING THESE NODAL FORCES TO THE SYSTEM OF EQUATIONS AT THE TOP OF PAGE 31 AND SOLVING GIVES

$$U_5 = -0.0010$$
$$U_6 = -0.4771$$
$$U_7 = 0.2616$$
$$U_8 = -0.4724$$
$$U_9 = 0.0059$$
$$U_{10} = -0.8970$$

3.17 $$\begin{bmatrix} 10 & -10 & 0 & 0 \\ -10 & 20 & -10 & 0 \\ 0 & -10 & 20 & -10 \\ 0 & 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ X_2 \\ 1.5 \\ X_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 20 \\ F_3 \\ 35 \end{Bmatrix}$$

IS THE SAME SYSTEM AS

$$\begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 20 & -10 & -10 \\ -10 & -10 & 20 & 0 \\ 0 & -10 & 0 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.5 \\ X_2 \\ X_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_3 \\ 20 \\ 35 \end{Bmatrix}$$

THE LOWER PARTITION (LAST TWO EQS.) IS

$$\begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_4 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 35 \end{Bmatrix} - \begin{bmatrix} -10 & -10 \\ 0 & -10 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.5 \end{Bmatrix}$$

HAVING SOLUTION

$$\begin{Bmatrix} X_2 \\ X_4 \end{Bmatrix} = \begin{bmatrix} \frac{1}{20} & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{Bmatrix} 20 \\ 35 \end{Bmatrix} - \begin{bmatrix} \frac{1}{20} & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} -10 & -10 \\ 0 & -10 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.5 \end{Bmatrix}$$

$$\begin{Bmatrix} X_2 \\ X_4 \end{Bmatrix} = \begin{Bmatrix} 1.75 \\ 5 \end{Bmatrix}$$

3.18

$$\begin{bmatrix} 50 & -50 & 0 & 0 \\ -50 & 100 & -50 & 0 \\ 0 & -50 & 75 & -25 \\ 0 & 0 & -25 & 25 \end{bmatrix} \begin{Bmatrix} U_1 \\ 0.5 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 30 \\ F_2 \\ 40 \\ 40 \end{Bmatrix}$$

IS EQUIVALENT TO

$$\begin{bmatrix} 50 & 0 & 0 \\ 0 & 75 & -25 \\ 0 & -25 & 25 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 30 \\ 40 \\ 40 \end{Bmatrix} - [-50 \quad -50 \quad 0]^T \{0.5\}$$

$$\begin{bmatrix} 50 & 0 & 0 \\ 0 & 75 & -25 \\ 0 & -25 & 25 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 55 \\ 65 \\ 40 \end{Bmatrix}$$

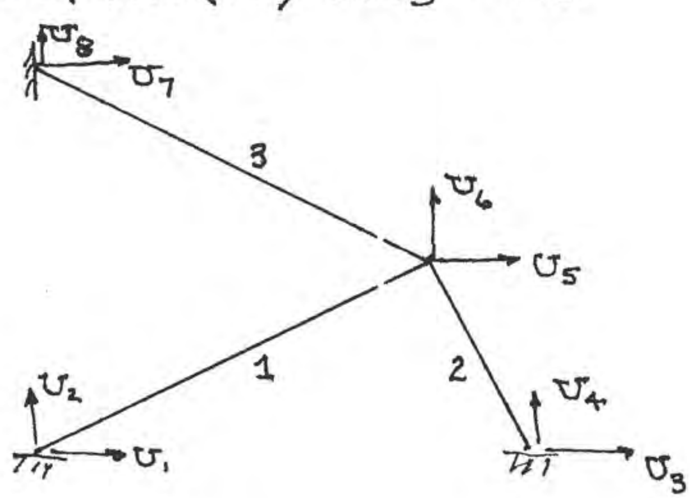
HAVING SOLUTION

$$U_1 = 1.1 \quad U_3 = 2.1 \quad U_4 = 3.7$$

AND

$$F_2 = -50U_1 + 100(0.5) - 50U_3 = 110$$

3.19



NODE	X	Y		ELEMENT	LENGTH (IN.)
1	0	0		1	72
2	83.13	0		2	41.57
3	62.35	36		3	72
4	0	72			

$$k^{(1)} = \frac{1\,(15)(10^6)}{72} = 2.08\,(10^5)\ \text{LB/IN} \qquad \theta_1 = 30^\circ$$

$$k^{(2)} = \frac{1\,(15)(10^6)}{41.57} = 3.61\,(10^5)\ \text{LB/IN} \qquad \theta_2 = 120^\circ$$

$$k^{(3)} = \frac{1\,(15)(10^6)}{72} = 2.08\,(10^5)\ \text{LB/IN} \qquad \theta_3 = -30^\circ$$

$$[K^{(1)}] = \begin{bmatrix} 15.63 & 9.02 & -15.63 & -9.02 \\ & 5.21 & -9.02 & -5.21 \\ & \text{SYM} & 15.63 & 9.02 \\ & & & 5.21 \end{bmatrix} (10^4)$$

$$[K^{(2)}] = \begin{bmatrix} 9.02 & -15.63 & 9.02 & 15.63 \\ & 27.07 & 15.62 & -27.07 \\ & \text{SYM} & 9.02 & -15.63 \\ & & & 27.07 \end{bmatrix} (10^4)$$

$$[K^{(3)}] = \begin{bmatrix} 15.63 & -9.02 & -15.63 & 9.02 \\ & 5.21 & 9.02 & -5.21 \\ & \text{SYM} & 15.63 & -9.02 \\ & & & 5.21 \end{bmatrix} (10^4)$$

$$[L_1] = [1 \quad 2 \quad 5 \quad 6]$$
$$[L_2] = [3 \quad 4 \quad 5 \quad 6]$$
$$[L_3] = [7 \quad 8 \quad 5 \quad 6]$$

ASSEMBLING ONLY THAT PORTION OF THE SYSTEM EQUATIONS CORRESPONDING TO NON-CONSTRAINED DISPLACEMENTS:

$$\begin{bmatrix} 40.28 & -15.63 \\ -15.63 & 37.49 \end{bmatrix} (10^4) \begin{Bmatrix} U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 4000 \\ 3000 \end{Bmatrix}$$

THE DISPLACEMENTS ARE

$$U_5 = 0.0155 \text{ in.} \qquad U_6 = 0.0145 \text{ in.}$$

ELEMENT STRESSES ARE:

$$\sigma^{(1)} = 15(10^6)\,\frac{U_5 \cos 30 + U_6 \sin 30}{72} = 4307 \text{ psi}$$

$$\sigma^{(2)} = 15(10^6)\,\frac{-U_5 \cos 60 + U_6 \sin 60}{41.57} = 1735 \text{ psi}$$

$$\sigma^{(3)} = 15(10^6)\,\frac{U_5 \cos 30 - U_6 \sin 30}{72} = 1286 \text{ psi}$$

AND ALL THREE ELEMENTS ARE IN TENSION.

3.20 $A = 2\ in^2 \quad E = 30\,(10^6)$

$$L^{(e)} = \left[(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2\right]^{1/2}$$

$$\cos\theta_x = \frac{X_j - X_i}{L^{(e)}} \quad \cos\theta_y = \frac{Y_j - Y_i}{L^{(e)}} \quad \cos\theta_z = \frac{Z_j - Z_i}{L^{(e)}}$$

$$[k^{(e)}] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[R] = \begin{bmatrix} \cos\theta_x & \cos\theta_y & \cos\theta_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_x & \cos\theta_y & \cos\theta_z \end{bmatrix}$$

$$[K^{(e)}] = [R]^T[k^{(e)}][R]$$

(a) $$[K] = \begin{bmatrix} 0.4243 & 0.3394 & 0.2546 & -0.4243 & -0.3394 & -0.2546 \\ 0.3394 & 0.2715 & 0.2036 & -0.3394 & -0.2715 & -0.2036 \\ 0.2546 & 0.2036 & 0.1527 & -0.2546 & -0.2036 & -0.1527 \\ -0.4243 & -0.3394 & -0.2546 & 0.4243 & 0.3394 & 0.2546 \\ -0.3394 & -0.2715 & -0.2036 & 0.3394 & 0.2715 & 0.2036 \\ -0.2546 & -0.2036 & -0.1527 & 0.2546 & 0.2036 & 0.1527 \end{bmatrix}(10^6)$$

(b) $$[K] = \begin{bmatrix} 0.5774 & 0.5774 & -0.5774 & -0.5774 & -0.5774 & 0.5774 \\ 0.5774 & 0.5774 & -0.5774 & -0.5774 & -0.5774 & 0.5774 \\ -0.5774 & -0.5774 & 0.5774 & 0.5774 & 0.5774 & -0.5774 \\ -0.5774 & -0.5774 & 0.5774 & 0.5774 & 0.5774 & -0.5774 \\ -0.5774 & -0.5774 & 0.5774 & 0.5774 & 0.5774 & -0.5774 \\ 0.5774 & 0.5774 & -0.5774 & -0.5774 & -0.5774 & 0.5774 \end{bmatrix}(10^6)$$

(c) $$[K^{(e)}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7500 & 0 & 0 & -0.7500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.7500 & 0 & 0 & 0.7500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} 10^6$$

(d) $$[K] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7500 & 0 & 0 & -0.7500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.7500 & 0 & 0 & 0.7500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} 10^6$$

THE MATRICES WERE COMPUTED VIA A SIMPLE MATLAB PROGRAM.

KMAT3D.M

```
x1=input('x1');
y1=input('y1');
z1=input('z1');
x2=input('x2');
y2=input('y2');
z2=input('z2');
e=30;
l=sqrt((x2-x1)^2+(y2-y1)^2+(z2-z1)^2);
cx=(x2-x1)/l;
cy=(y2-y1)/l;
cz=(z2-z1)/l;
a=2;
k=a*e/l;
r=[cx cy cz 0 0 0;0 0 0 cx cy cz];
k1=[1 -1;-1 1];
ke=k*r'*k1*r;
ke
l
```

3.21

$$[K^{(e)}] = k^{(e)} \begin{bmatrix} C_x & 0 \\ C_y & 0 \\ C_z & 0 \\ 0 & C_x \\ 0 & C_y \\ 0 & C_z \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_X & C_Y & C_Z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_X & C_Y & C_Z \end{bmatrix}$$

$$= k^{(e)} \begin{bmatrix} C_x & 0 \\ C_y & 0 \\ C_z & 0 \\ 0 & C_x \\ 0 & C_y \\ 0 & C_z \end{bmatrix} \begin{bmatrix} C_X & C_Y & C_Z & -C_X & -C_Y & -C_Z \\ -C_X & -C_Y & -C_Z & C_X & C_Y & C_Z \end{bmatrix}$$

$$= k^{(e)} \begin{bmatrix} C_X^2 & C_XC_Y & C_XC_Z & -C_X^2 & -C_XC_Y & -C_XC_Z \\ & C_Y^2 & C_YC_Z & -C_YC_X & -C_Y^2 & -C_YC_Z \\ & & C_Z^2 & -C_XC_Z & -C_YC_Z & -C_Z^2 \\ & \text{SYM} & & C_X^2 & C_XC_Y & C_XC_Z \\ & & & & C_Y^2 & C_YC_Z \\ & & & & & C_Z^2 \end{bmatrix}$$

3.22

$$u(x) = N_1 u_1 + N_2 u_2$$

$$\epsilon = \frac{\partial u}{\partial x} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2$$

$$\epsilon = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\epsilon = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} [R] \begin{Bmatrix} U^{(e)} \end{Bmatrix}$$

$$\epsilon = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} [R] \{U^{(e)}\}$$

$$\sigma = E\epsilon = E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} [R] \{U^{(e)}\}$$

3.23

$U_1 = U_2 = U_3 = 0 \quad U_4 = U_5 = U_6 = 0.06$

USING THE GEOMETRIC DATA AND THE STRESS EQUATION GIVEN IN PROBLEM 3.22 IN A SHORT MATLAB PROGRAM GIVES

(a) $\sigma = 43{,}200$ psi
(b) $\sigma = 30{,}000$ psi
(c) $\sigma = 22{,}500$ psi
(d) $\sigma = -22{,}500$ psi

3.24

$$U_e = \frac{1}{2} k^{(e)} (u_2 - u_1)^2$$

$$= \frac{1}{2} k^{(e)} \left(U_4 cx + U_5 cy + U_6 cz - U_1 cx - U_2 cy - U_3 cz \right)^2$$

WHERE $cx = \cos\theta_x$ $cy = \cos\theta_y$ $cz = \cos\theta_z$

$$U_e = \frac{1}{2} k^{(e)} \left(-U_1 cx - U_2 cy - U_3 cz + U_4 cx + U_5 cy + U_6 cz \right)^2$$

$$\frac{\partial U_e}{\partial U_1} = k^{(e)} \left(-U_1 cx - U_2 cy - U_3 cz + U_4 cx + U_5 cy + U_6 cz \right)(-cx)$$
$$= F_1$$

OR

$$k^{(e)} \left(cx^2 U_1 + cx\,cy\,U_2 + cx\,cz\,U_3 - cx^2 U_4 - cx\,cy\,U_5 - cx\,cz\,U_6 \right) = F_1$$

THE COEFFICIENTS OF THE DISPLACEMENTS IN THE EQUATION ABOVE ARE IDENTICAL TO THE TERMS IN THE FIRST ROW OF THE STIFFNESS MATRIX IN EQ. 3.58.

TAKING THE REMAINING PARTIAL DERIVATIVES

$$\frac{\partial U_e}{\partial U_i} = F_i \qquad i = 2,6$$

WILL VERIFY THE REMAINER OF THE STIFFNESS MATRIX.

3.25

VERIFICATION OF THE STIFFNESS MATRIX VIA MINIMUM POTENTIAL ENERGY REQUIRES FORMULATION OF THE TOTAL POTENTIAL ENERGY FUNCTION

$$\Pi_P = \tfrac{1}{2}k^{(e)}\left(U_4 c_X + U_5 c_Y + U_6 c_Z - U_1 c_X - U_2 c_Y - U_3 c_Z\right)^2 - F_1U_1 - F_2U_2 - F_3U_3 - F_4U_4 - F_5U_5 - F_6U_6$$

AND EXAMINING THE RELATIONS

$$\frac{\partial \Pi_P}{\partial U_i} = 0 \qquad i = 1, 6$$

THE RESULTING STIFFNESS MATRIX IS IDENTICAL TO EQ. 3.58.

3.26

ELEMENT	NODES	$L^{(e)}$	$\cos\theta_X$	$\cos\theta_Y$	$\cos\theta_Z$
1	1–4	43.87	0.684	−0.456	0.570
2	2–4	36.40	0.824	−0.549	−0.137
3	3–4	33.54	−0.298	−0.596	0.745

$$k^{(1)} = \frac{1.5(10)10^6}{43.87} = 3.42(10^5) \text{ LB/IN.}$$

$$k^{(2)} = \frac{1.5\,(10)(10^6)}{36.4} = 4.12\,(10^5)\ \mathrm{LB/IN}$$

$$k^{(3)} = \frac{1.5\,(10)(10^6)}{33.54} = 4.47\,(10^5)\ \mathrm{LB/IN.}$$

TRANSFORMING TO GLOBAL COORDINATES, THE ELEMENT STIFFNESS MATRICES ARE:

ELEMENT 1

ke =

$$\begin{bmatrix} 0.1598 & -0.1066 & 0.1332 & -0.1598 & 0.1066 & -0.1332 \\ -0.1066 & 0.0710 & -0.0888 & 0.1066 & -0.0710 & 0.0888 \\ 0.1332 & -0.0888 & 0.1110 & -0.1332 & 0.0888 & -0.1110 \\ -0.1598 & 0.1066 & -0.1332 & 0.1598 & -0.1066 & 0.1332 \\ 0.1066 & -0.0710 & 0.0888 & -0.1066 & 0.0710 & -0.0888 \\ -0.1332 & 0.0888 & -0.1110 & 0.1332 & -0.0888 & 0.1110 \end{bmatrix} 10^6$$

ELEMENT 2

ke =

$$\begin{bmatrix} 0.2799 & -0.1866 & -0.0467 & -0.2799 & 0.1866 & 0.0467 \\ -0.1866 & 0.1244 & 0.0311 & 0.1866 & -0.1244 & -0.0311 \\ -0.0467 & 0.0311 & 0.0078 & 0.0467 & -0.0311 & -0.0078 \\ -0.2799 & 0.1866 & 0.0467 & 0.2799 & -0.1866 & -0.0467 \\ 0.1866 & -0.1244 & -0.0311 & -0.1866 & 0.1244 & 0.0311 \\ 0.0467 & -0.0311 & -0.0078 & -0.0467 & 0.0311 & 0.0078 \end{bmatrix} 10^6$$

ELEMENT 3

ke =

$$\begin{bmatrix} 0.0398 & 0.0795 & -0.0994 & -0.0398 & -0.0795 & 0.0994 \\ 0.0795 & 0.1590 & -0.1988 & -0.0795 & -0.1590 & 0.1988 \\ -0.0994 & -0.1988 & 0.2485 & 0.0994 & 0.1988 & -0.2485 \\ -0.0398 & -0.0795 & 0.0994 & 0.0398 & 0.0795 & -0.0994 \\ -0.0795 & -0.1590 & 0.1988 & 0.0795 & 0.1590 & -0.1988 \\ 0.0994 & 0.1988 & -0.2485 & -0.0994 & -0.1988 & 0.2485 \end{bmatrix} 10^6$$

ELEMENT TO GLOBAL DISPLACEMENT RELATIONS ARE:

$$[L_1] = [\,1 \quad 2 \quad 3 \quad 10 \quad 11 \quad 12\,]$$

$$[L_2] = [\,4 \quad 5 \quad 6 \quad 10 \quad 11 \quad 12\,]$$

$$[L_3] = [\,7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12\,]$$

ASSEMBLING ONLY THE PORTION OF THE GLOBAL SYSTEM CORRESPONDING TO NONZERO DISPLACEMENTS U_{10}, U_{11}, U_{12}:

$$10^6 \begin{bmatrix} 0.4795 & -0.2137 & -0.3129 \\ & 0.3544 & -0.2565 \\ \text{SYM} & & 0.3673 \end{bmatrix} \begin{Bmatrix} U_{10} \\ U_{11} \\ U_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1500 \\ 0 \end{Bmatrix}$$

THE DISPLACEMENTS ARE

$$U_{10} = -0.0097 \text{ IN.}$$

$$U_{11} = -0.0209 \text{ IN.}$$

$$U_{12} = -0.0149 \text{ IN.}$$

ELEMENT STRESSES (SEE PROBLEM 3.22) ARE

$$\sigma^{(1)} = -1275 \text{ psi}$$

$$\sigma^{(2)} = 1521 \text{ psi}$$

$$\sigma^{(3)} = 1267 \text{ psi}$$

CHAPTER 4

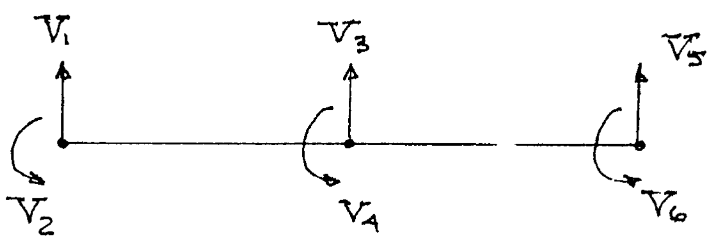
4.1 THE BENDING STRESS AT NODE 2 OF ELEMENT 1 IS GIVEN BY EQ. 4.32 WITH $x = L$ AS

$$\sigma_x = y_{max} E \left[\frac{6}{L^2}\left(v_1^{(1)} - v_2^{(1)}\right) + \frac{2}{L}\left(2\theta_2^{(1)} + \theta_1^{(1)}\right) \right] \quad (1)$$

FOR ELEMENT 2 THE STRESS IS GIVEN BY EQ. 4.32 WITH $x = 0$ AS

$$\sigma_x = y_{max} E \left[\frac{6}{L^2}\left(v_2^{(2)} - v_1^{(2)}\right) - \frac{2}{L}\left(2\theta_1^{(2)} + \theta_2^{(2)}\right) \right] \quad (2)$$

DENOTE THE GLOBAL DISPLACEMENTS AS SHOWN BELOW



AND WRITE

$$V_1 = v_1^{(1)}$$
$$V_2 = \theta_1^{(1)}$$
$$V_3 = v_2^{(1)} = v_1^{(2)}$$
$$V_4 = \theta_2^{(1)} = \theta_1^{(2)}$$
$$V_5 = v_2^{(2)}$$
$$V_6 = \theta_2^{(2)}$$

SUBSTITUTE THE GLOBAL DISPLACEMENTS INTO THE TWO STRESS EQUATIONS TO OBTAIN:

$$\frac{6}{L^2}(V_1 - V_3) + \frac{2}{L}(2V_4 + V_2)$$

$$= \frac{6}{L^2}(V_5 - V_3) - \frac{2}{L}(2V_4 + V_6) \quad (3)$$

EQUALITY WILL HOLD ONLY IF THE STRESS IS CONTINUOUS ACROSS/AT THE ELEMENT BOUNDARY.

FOR EQUALITY

$$\frac{6}{L^2}(V_1 - V_5) + \frac{2}{L}(4V_4 + V_2 + V_6) = 0 \quad (4)$$

CLEARLY THIS EXPRESSION IS NOT TRUE, IN GENERAL. (NUMERICALLY, ZERO VALUE IS POSSIBLE BUT NOT LIKELY.) ONE PHYSICAL CASE IN WHICH (4) HOLDS IS:

$$V_1 = V_5 \quad (5)$$

$$V_4 = 0 \quad (6)$$

$$V_6 = -V_2 \quad (7)$$

(5) STATES THAT NODES 1 AND 3 HAVE THE SAME TRANSVERSE DEFLECTION. WITH (6) AND (7) THE SITUATION IS SYMMETRIC BENDING AROUND NODE 2.

4.2 THE ELEMENT IS LOADED ONLY AT THE NODES (AS REQUIRED) THUS SHEAR FORCE IS CONSTANT OVER THE LENGTH AND BENDING MOMENT VARIES LINEARLY. EQ. 4.10 IS THEN

$$EI_z \frac{d^2v}{dx^2} = C_1 + C_2 x$$

AND EQ. 4.17 GIVES

$$\frac{d^2v}{dx^2} = 2a_2 + 6a_3 x$$

SO THE NODAL-ONLY LOAD RESTRICTION AND THE ASSUMED DISPLACEMENT FUNCTION ARE IN ACCORD.

4.3 THE EQUIVALENT NODAL LOADS ARE

$$\{F\} = \begin{Bmatrix} -qL/2 \\ -qL^2/12 \\ -qL/2 \\ qL^2/12 \end{Bmatrix}$$

AND THE CONSTRAINT CONDITIONS ARE $V_1 = V_2 = 0$ SO WE MUST SOLVE FOR THE NODAL SLOPES AND COMPUTE MIDSPAN DEFLECTION VIA EQ. 4.26.

$$\frac{EI_z}{L^3}\begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix}\theta_1 \\ \theta_2\end{Bmatrix} = \frac{qL^2}{12}\begin{Bmatrix}-1 \\ 1\end{Bmatrix}$$

OR

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix}\theta_1 \\ \theta_2\end{Bmatrix} = \frac{qL^3}{24EI_z}\begin{Bmatrix}-1 \\ 1\end{Bmatrix}$$

WHICH GIVES

$$\theta_1 = -\frac{qL^3}{24EI_z} \qquad \theta_2 = \frac{qL^3}{24EI_z}$$

WITH $v_1 = v_2 = 0$, EQ. 4.26 IS

$$v(x) = \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right)\theta_1 + \left(\frac{x^3}{L^2} - \frac{x^2}{L}\right)\theta_2$$

SUBSTITUTING FOR θ_1 AND θ_2 WITH $x = \frac{L}{2}$

$$v\left(\frac{L}{2}\right) = -\frac{qL^4}{96EI_z} = -\frac{4qL^4}{384EI_z}$$

SO USING ONLY ONE ELEMENT RESULTS IN A 20% ERROR.

4.4 $\quad q(x) = -q_0 \frac{x}{L}$

$$F_{1q} = \int_0^L \left(-q_0 \frac{x}{L}\right)\left(1 - 3\frac{x^2}{L^2} + \frac{2x^3}{L^3}\right) dx$$

$$= -q_0 \left[\frac{x^2}{2L} - \frac{3x^4}{4L^3} + \frac{2x^5}{5L^4} \right|_0^L = -\frac{3q_0 L}{20}$$

$$M_{1q} = \int_0^L \left(-q_0 \frac{x}{L}\right)\left(x - 2\frac{x^2}{L} + \frac{x^3}{L^2}\right) dx$$

$$= -q_0 \left[\frac{x^3}{3L} - \frac{x^4}{2L^2} + \frac{x^5}{5L^3} \right|_0^L = -\frac{q_0 L^2}{30}$$

$$F_{2q} = \int_0^L \left(-q_0 \frac{x}{L}\right)\left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right) dx$$

$$= -q_0 \left[\frac{3x^4}{4L^3} - \frac{2x^5}{5L^4} \right|_0^L = -\frac{7q_0 L}{20}$$

$$M_{2q} = \int_0^L \left(-q_0 \frac{x}{L}\right)\left(\frac{x^3}{L^2} - \frac{x^2}{L}\right) dx$$

$$= -q_0 \left[\frac{x^5}{5L^3} - \frac{x^4}{4L^2} \right|_0^L = \frac{q_0 L^2}{20}$$

4.5

$$\frac{EI_z}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{q_0 L}{20}\begin{Bmatrix} -7 \\ L \end{Bmatrix}$$

OR

$$\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{q_0 L^4}{40 EI_z}\begin{Bmatrix} -7 \\ L \end{Bmatrix}$$

VIA CRAMER'S RULE

$$v_2 = -\frac{11 q_0 L^4}{120 EI_z}$$

$$\theta_2 = -\frac{q_0 L^3}{8 EI_z}$$

4.6 THE CONSTRAINT EQUATIONS ARE

$$\frac{EI_z}{L^3}\begin{bmatrix} -12 & 6L \\ -6L & 2L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} R_1 - \frac{3q_0 L}{20} \\ M_{R1} - \frac{q_0 L^2}{30} \end{Bmatrix}$$

$$R_1 = \frac{3q_0L}{20} + \frac{12EI_z}{L^3}\left(\frac{11q_0L^4}{120EI_z}\right) - \frac{6EI_z}{L^2}\left(\frac{q_0L^3}{8EI_z}\right)$$

$$R_1 = \frac{q_0L}{2}$$

$$M_{R1} = \frac{q_0L^2}{30} + \frac{6EI_z}{L^2}\left(\frac{11q_0L^4}{120EI_z}\right) - \frac{2EI_z}{L}\left(\frac{q_0L^3}{8EI_z}\right)$$

$$M_{R1} = \frac{q_0L^2}{3}$$

MAXIMUM STRESS OCCURS AT NODE 1. PER EQ. 4.33:

$$\sigma_x = \pm hE\left(\frac{6}{L^2}v_2 - \frac{2}{L}\theta_2\right)$$

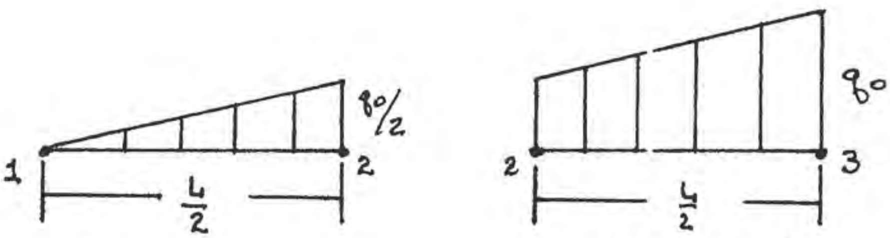
$$= \pm hE\left[\frac{6}{L^2}\left(-\frac{11q_0L^4}{120EI_z}\right) - \frac{2}{L}\left(-\frac{q_0L^3}{8EI_z}\right)\right]$$

$$= \pm\frac{3q_0L^2h}{10I_z}$$

STRENGTH OF MATERIALS:

$$\sigma_{max} = \pm\frac{\left(\frac{q_0L^2}{3}\right)h}{I_z} = \pm\frac{q_0L^2h}{3}$$

4.7 USE OF TWO ELEMENTS REQUIRES THAT THE NODAL LOADS BE RECALCULATED SINCE WE NOW HAVE:



ELEMENT 1 ELEMENT 2

ELEMENT 1 (USING THE RESULTS OF PROBLEM 4.4)

$$F_{1q}^{(1)} = -\frac{3\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)}{20} = -\frac{3q_0L}{80}$$

$$M_{1q}^{(1)} = -\frac{\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)^2}{30} = -\frac{q_0L^2}{240}$$

$$F_{2q}^{(1)} = -\frac{7\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)}{20} = -\frac{7q_0L}{80}$$

$$M_{2q}^{(1)} = \frac{\frac{q_0}{2}\left(\frac{L}{2}\right)^2}{20} = \frac{q_0L^2}{160}$$

ELEMENT 2 HERE WE USE SUPERPOSITION WITH A UNIFORM LOAD $q_0/2$ AND A LINEAR LOAD $q_0/2$ TO OBTAIN

$$F_{1q}^{(2)} = -\frac{\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)}{2} - \frac{3\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)}{20} = -\frac{13q_0L}{80}$$

$$M_{1q}^{(2)} = -\frac{\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)^2}{12} - \frac{\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)^2}{30} = -\frac{7 q_0 L^2}{480}$$

$$F_{2q}^{(2)} = -\frac{\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)}{2} - \frac{7\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)}{20} = -\frac{17 q_0 L}{80}$$

$$M_{2q}^{(2)} = \frac{\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)^2}{12} + \frac{\left(\frac{q_0}{2}\right)\left(\frac{L}{2}\right)^2}{20} = \frac{8 q_0 L^2}{45}$$

THE ELEMENT STIFFNESS MATRICES ARE IDENTICAL AND GIVEN BY (USING $L_e = L/2$)

$$[K^{(1)}] = [K^{(2)}] = \frac{8EI_z}{L^3}\begin{bmatrix} 12 & 3L & -12 & 3L \\ & L & -3 & L/2 \\ & & 12 & -3L \\ \text{SYM} & & & L \end{bmatrix}$$

THE ASSEMBLED (6×6) GLOBAL STIFFNESS MATRIX IS

$$[K] = \frac{8EI_z}{L^3}\begin{bmatrix} 12 & 3L & -12 & 3L & 0 & 0 \\ & L & -3 & L/2 & 0 & 0 \\ & & 24 & 0 & -12 & 3L \\ & & & 2L & -3 & L/2 \\ \text{SYM} & & & & 12 & -3L \\ & & & & & L \end{bmatrix}$$

APPLYING THE CONSTRAINT CONDITIONS $U_1 = U_2 = 0$ AND THE NODAL FORCES:

$$\frac{EI_z}{L^3}\begin{bmatrix} 24 & 0 & -12 & 3L \\ & 2L & -3 & L/2 \\ & & 12 & -3L \\ \text{SYM} & & & L \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = -q_0 L \begin{Bmatrix} 1/4 \\ L/120 \\ 17/80 \\ -8L/45 \end{Bmatrix}$$

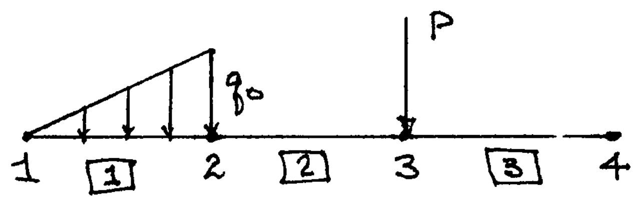
SUBSTITUTING $E = 30(10^6)$, $I_z = 0.1$, $L = 10$, $q_0 = 10$ WE HAVE:

$$3(10^3)\begin{bmatrix} 24 & 0 & -12 & 30 \\ & 20 & -3 & 5 \\ & & 12 & -30 \\ \text{SYM} & & & 10 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} -25.00 \\ -8.33 \\ -21.25 \\ 177.78 \end{Bmatrix}$$

WHICH HAS THE SOLUTION

$$U_3 = -1.42(10^{-3}) \text{ IN.}$$
$$U_4 = -5.38(10^{-4}) \text{ RAD}$$
$$U_5 = -3.69(10^{-3}) \text{ IN.}$$
$$U_6 = -6.19(10^{-4}) \text{ RAD.}$$

4.8 SINCE THE BEAM ELEMENT IS RESTRICTED TO LOADING AT THE NODES ONLY, THIS BEAM CAN BE MODELED BY NO FEWER THAN THREE ELEMENTS. THE NODES MUST BE LOCATED, IN ORDER, (1) AT THE LEFT SUPPORT, (2) AT L/3, THE END OF THE LINEARLY-VARYING LOAD, (3) AT THE POINT OF THE CONCENTRATED LOAD P, (4) AT THE RIGHT-HAND SUPPORT.



THE NODAL LOAD VECTOR WILL INCLUDE BOTH APPLIED FORCE AND MOMENT AT NODES 1 AND 2, APPLIED LOAD AT NODE 3, AND APPLIED LOAD (REACTION FORCE) AT NODE 4.

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \\ F_4 \\ M_4 \end{Bmatrix} = \begin{Bmatrix} R_1 - 3q_0L/60 \\ -q_0L^2/270 \\ -7q_0L/60 \\ q_0L^2/180 \\ -P \\ 0 \\ R_4 \\ 0 \end{Bmatrix}$$

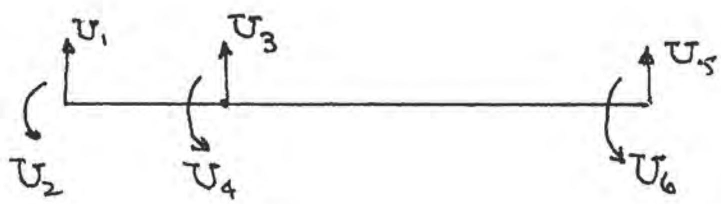
4.9 AT A SPECIFIED X POSITION ALONG THE BEAM AXIS, CURVATURE OF THE NEUTRAL AXIS AS APPROXIMATED BY $\frac{d^2v}{dx^2}$ DOES NOT VARY OVER THE CROSS SECTION.

4.10 $I_z = \frac{bh^3}{12} = \frac{(1)(2)^3}{12} = \frac{2}{3} \text{ in}^4$

TWO ELEMENTS $L^{(1)} = 10$ in. $L^{(2)} = 30$ in.

$$[K^{(1)}] = \frac{10(10^6)(\frac{2}{3})}{10^3}\begin{bmatrix} 12 & 60 & -12 & 60 \\ & 400 & -60 & 200 \\ & \text{SYM} & 12 & -60 \\ & & & 400 \end{bmatrix}$$

$$[K^{(2)}] = \frac{10(10^6)(\frac{2}{3})}{9(10^3)}\begin{bmatrix} 12 & 180 & -12 & 180 \\ & 3600 & -180 & 1800 \\ & \text{SYM} & 12 & -180 \\ & & & 3600 \end{bmatrix}$$



THE ASSEMBLED GLOBAL EQUATIONS ARE

$$\frac{2}{3}(10^4)\begin{bmatrix} 12 & 60 & -12 & 60 & 0 & 0 \\ & 400 & -60 & 200 & 0 & 0 \\ & & 40/3 & -40 & -4/3 & 20 \\ & & & 800 & -20 & 200 \\ & \text{SYM} & & & 4/3 & -20 \\ & & & & & 400 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}$$

$$= \begin{Bmatrix} R_1 \\ 0 \\ -500 \\ 0 \\ R_5 \\ 0 \end{Bmatrix}$$

SETTING $U_1 = U_5 = 0$ AND SOLVING THE REMAINING 4 x 4 SYSTEM GIVES

$$U_2 = -1.67(10^{-3}) \text{ RAD}$$

$$U_3 = -1.25(10^{-2}) \text{ IN.}$$

$$U_4 = -4.2(10^{-3}) \text{ RAD}$$

$$U_6 = 8.33(10^{-4}) \text{ RAD}$$

MAXIMUM STRESS IS COMPUTED FOR EACH ELEMENT USING EQS. 4.33 AND 4.34

ELEMENT 1

$$\sigma_x(x=L) = (1)(10)(10^6)\left[\frac{6}{10^2}\left(0 + 1.25(10^{-2})\right) + \frac{2}{10}\left(2(-4.2) - 1.65\right)10^{-3}\right] = \mp 12{,}640 \text{ psi}$$

ELEMENT 2

$$\sigma_x(x=0) = (1)(10)(10^6)\left[\frac{6}{30^2}(1.25)(10^{-2}) - \frac{2}{30}\left[-(2)(4.2) + 0.833\right]10^{-3}\right]$$

$$= 5873 \text{ psi}$$

4.11 TWO ELEMENTS (MINIMUM) ARE REQUIRED WITH ELEMENT LENGTH = 10 IN. THUS THE ELEMENT STIFFNESS MATRICES ARE IDENTICAL.

$$I_z = \frac{bh^3}{12} = \frac{0.5(0.5)^3}{12} = 5.21(10^{-3}) \text{ in}^4$$

$$\frac{EI_z}{L^3} = \frac{10(10^6)(5.21)(10^{-3})}{10^3} = 52.1 \text{ lb/in}$$

$$[K^{(1)}] = [K^{(2)}] = 52.1 \begin{bmatrix} 12 & 60 & -12 & 60 \\ 60 & 400 & -60 & 200 \\ -12 & -60 & 12 & -60 \\ 60 & 200 & -60 & 400 \end{bmatrix}$$

FOR ELEMENT 1, THE NODAL FORCE VECTOR IS:

$$\{f\} = \begin{Bmatrix} -50 \\ -83.33 \\ -50 \\ 83.33 \end{Bmatrix}$$

THE ASSEMBLED SYSTEM EQUATIONS ARE

$$52.1 \begin{bmatrix} 12 & 60 & -12 & 60 & 0 & 0 \\ & 400 & -60 & 200 & 0 & 0 \\ & & 24 & 0 & -12 & 60 \\ & & & 800 & -60 & 200 \\ & \text{SYM} & & & 12 & -60 \\ & & & & & 400 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} R_1 - 50 \\ -83.33 \\ -50 \\ 83.33 \\ R_5 \\ 0 \end{Bmatrix}$$

APPLYING THE CONSTRAINT CONDITIONS $U_1 = U_5 = 0$ AND SOLVING THE RESULTING 4×4 SYSTEM GIVES

$$U_2 = -0.036 \text{ RAD}$$
$$U_3 = -0.200 \text{ IN.}$$
$$U_4 = 0.004 \text{ RAD.}$$
$$U_5 = 0.028 \text{ RAD.}$$

4.12

TWO ELEMENTS (MINIMUM) ARE REQUIRED; ONE WITH UNIFORM LOAD, ONE WITH LINEAR LOAD.

CALCULATE SECTION'S PROPERTIES

$$\bar{y}A = \bar{y}_1 A_1 + \bar{y}_2 A_2$$
$$\bar{y}\left[40(10) + 30(10)\right] = 15(10)(30) + 35(10)(40)$$
$$\bar{y} = 26.429 \text{ mm.}$$

$$I_z = \frac{10(30)^3}{12} + (11.429)^2(10)(30) + \frac{40(10)^3}{12} + (8.571)^2(40)(10)$$

$$I_z \cong 94.4(10^3) \text{ mm}^4 \qquad E = 10(10^6) \text{ psi} \cong 69(10^3) \text{ MPa}$$

$$\frac{EI_z}{L^3} = \frac{69(10^3)(94.4)(10^3)}{300^3} = 241.2 \text{ N/mm} \qquad L = 300 \text{ mm}$$

$$[K^{(1)}] = [K^{(2)}] = 241.2 \begin{bmatrix} 12 & 1800 & -12 & 1800 \\ & 360000 & -1800 & 180000 \\ & & 12 & -1800 \\ \text{SYM} & & & 360000 \end{bmatrix}$$

THE LOAD VECTOR FOR ELEMENT 1 USING $q = 4.5\ (10^{-2})$ N/mm AND $L = 300$ mm IS

$$\{f^{(1)}\} = \begin{Bmatrix} -6.75 \\ -337.5 \\ -6.75 \\ +337.5 \end{Bmatrix}$$

FOR ELEMENT 2, THE RESULT OF PROBLEM 4.4 GIVES THE LOAD VECTOR AS

$$\{f^{(2)}\} = \begin{Bmatrix} -4.725 \\ -202.5 \\ -2.025 \\ 135 \end{Bmatrix}$$

THUS

$$241.2 \begin{bmatrix} 12 & 1800 & -12 & 1800 & 0 & 0 \\ & 360000 & -1800 & 180000 & 0 & 0 \\ & & 24 & 0 & -12 & 1800 \\ & \text{SYM} & & 720000 & -1800 & 180000 \\ & & & & 12 & -1800 \\ & & & & & 360000 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} R_1 - 6.75 \\ -337.5 \\ -11.475 \\ 135 \\ -2.025 \\ 135 \end{Bmatrix}$$

APPLYING CONSTRAINT CONDITIONS $U_1 = U_5 = 0$, WE OBTAIN

$$U_2 = -5.26\,(10^{-5}) \text{ RAD}$$
$$U_3 = -0.0096 \text{ MM}$$
$$U_4 = 1.81\,(10^{-6}) \text{ RAD}$$
$$U_6 = 4.85\,(10^{-5}) \text{ RAD}$$

STRESSES: $\quad y_{MAX} = -26.429$ MM

ELEMENT 1

$$\sigma_{MAX} = -26.429\,(69)(10^3)\left[\frac{6}{300^2}(0.0096) + \frac{2}{300}\left\{2\,(1.81)(10^{-6}) - 5.26\,(10^{-5})\right\}\right] = -0.57 \text{ MPa}$$

ELEMENT 2

$$\sigma_{MAX} = -26.429\,(69)(10^3)\left[\frac{6}{300^2}(-0.0096) - \frac{2}{300}\left\{2\,(1.81)(10^{-6}) + 4.85\,(10^{-5})\right\}\right] = 1.8 \text{ MPa}$$

$\underline{4.13}$ $I_z = \dfrac{2(1)^3}{12} = 0.1667 \text{ in}^4$

TWO BEAM ELEMENTS AND A LINEAR SPRING ARE REQUIRED.

<u>ELEMENT 1</u> $L = 18''$ $\dfrac{EI_z}{L^3} = \dfrac{10(10^6)(0.1667)}{18^3} = 2.86(10^2)$

$$[K^{(1)}] = 2.86(10^2)\begin{bmatrix} 12 & 108 & -12 & 108 \\ & 1296 & -108 & 648 \\ & & 12 & -108 \\ \text{SYM} & & & 1296 \end{bmatrix}$$

<u>ELEMENT 2</u> $L = 6''$ $\dfrac{EI_z}{L^3} = \dfrac{10(10^6)(0.1667)}{6^3} = 7.72(10^3)$

$$[K^{(2)}] = 7.72(10^3)\begin{bmatrix} 12 & 36 & -12 & 36 \\ & 144 & -36 & 72 \\ \text{SYM} & & 12 & -36 \\ & & & 144 \end{bmatrix}$$

THE SPRING STIFFNESS ADDS DIRECTLY TO THE STIFFNESS AT THE RIGHT-HAND NODE OF ELEMENT 2.

THE ASSEMBLED GLOBAL STIFFNESS MATRIX IS:

$$\begin{bmatrix} 3.432 & 30.89 & -3.432 & 30.89 & 0 & 0 \\ & 370.7 & -30.89 & 185.3 & 0 & 0 \\ & & 96.07 & 247.0 & -92.64 & 277.9 \\ & & & 1482 & -277.9 & 555.8 \\ & \text{SYM} & & & 92.64 & -277.9 \\ & & & & & 1111.7 \end{bmatrix}(10^3)$$

THE NODAL FORCE VECTOR IS

$$\{F\} = \begin{Bmatrix} R_1 \\ 0 \\ -200 \\ 0 \\ -400U_5 \\ 0 \end{Bmatrix}$$

GIVEN THE ELASTIC SUPPORT, THE ONLY CONSTRAINT IS $U_1 = 0$. ELIMINATING THE CONSTRAINT AND SOLVING THE REMAINING 5×5 SYSTEM GIVES:

$U_2 = -0.018$ RAD.

$U_3 = -0.302$ IN

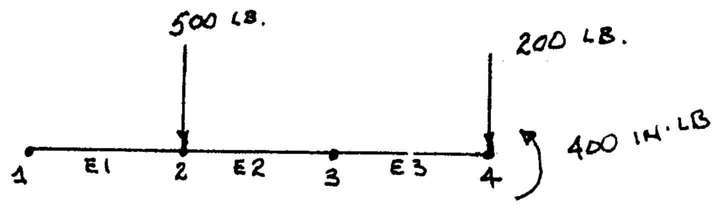
$U_4 = -0.013$ RAD

$U_5 = -0.377$ IN

$U_6 = -0.012$ RAD

NOTE THAT AS A RESULT OF THE SOFT SPRING SUPPORT, THE BEAM ESSENTIALLY ROTATES AS A RIGID BODY.

4.14 THREE ELEMENTS (MINIMUM) ARE REQUIRED AS SHOWN:



ELEMENT 1 $\quad L^{(1)} = 10$ IN. $\quad D = 1.5$ IN.

$$I_z = \frac{\pi}{64}(1.5)^4 = 0.249 \text{ in}^4$$

$$\frac{EI_z}{L^3} = 2.49(10^3)$$

$$[K^{(1)}] = \begin{bmatrix} 29.88 & 149.4 & -29.88 & 149.4 \\ & 996 & -149.4 & 498 \\ & \text{SYM} & 29.88 & -149.4 \\ & & & 996 \end{bmatrix}(10^3)$$

ELEMENT 2 $\quad L^{(2)} = 8$ IN. $\quad D = 1.5$ IN.

$$\frac{EI_z}{L^3} = 4.86(10^3)$$

$$[K^{(2)}] = \begin{bmatrix} 58.32 & 233.3 & -58.32 & 233.3 \\ & 1244 & -233.3 & 622 \\ & \text{SYM} & 58.32 & -233.3 \\ & & & 1244 \end{bmatrix} (10^3)$$

ELEMENT 3

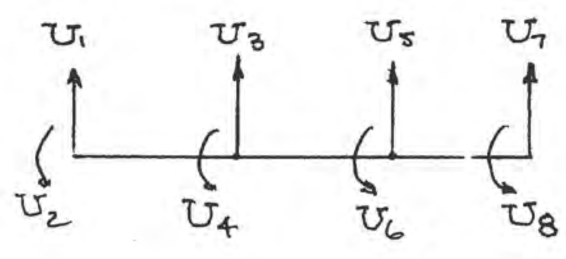
$L^{(3)} = 8$ IN. $\quad D = 1$ IN

$$I_z = \frac{\pi}{64}(1)^4 = 0.049 \text{ IN}^4$$

$$\frac{EI_z}{L^3} = 0.957(10^3)$$

$$[K^{(3)}] = \begin{bmatrix} 11.48 & 45.94 & -11.48 & 45.94 \\ & 245 & -45.94 & 122.5 \\ & \text{SYM} & 11.48 & -45.94 \\ & & & 245 \end{bmatrix} (10^3)$$

USING THE DISPLACEMENT NOTATION SHOWN:



AND THE CONSTRAINT CONDITIONS $U_1 = U_2 = 0$

RESULTS IN

$$
\begin{bmatrix}
88.2 & 83.9 & -58.32 & 233.3 & 0 & 0 \\
 & 2240 & -233.3 & 622 & 0 & 0 \\
 & & 69.8 & -187.4 & -11.48 & 45.94 \\
 & & & 1489 & -45.94 & 122.5 \\
 & \text{SYM} & & & 11.48 & -45.94 \\
 & & & & & 245
\end{bmatrix}
\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}
$$

$$
= \begin{Bmatrix} -500 \\ 0 \\ 0 \\ 0 \\ -200 \\ 400 \end{Bmatrix} (10^{-3})
$$

THE DISPLACEMENT SOLUTION IS

$U_3 = -0.152$ IN.

$U_4 = -0.026$ RAD

$U_5 = -0.386$ IN.

$U_6 = -0.032$ RAD.

$U_7 = -0.688$ IN.

$U_8 = -0.039$ RAD.

$\sigma_{MAX} = \pm 29400$ psi AT NODE 1

4.15 THREE ELEMENTS (MINIMUM) ARE REQUIRED.

$L^{(1)} = L^{(3)} = 10$ IN. $\quad L^{(2)} = 20$ IN.

LOCATE CENTROID:

$$\bar{y}\left[0.25(1.75)(2) + 0.25(2)\right] = 2\left(\frac{1.75}{2}\right)(0.25)(1.75) + 1.875(0.25)(2)$$

$$\bar{y} = 1.239 \text{ IN.}$$

CALCULATE I_z VIA PARALLEL AXIS THEOREM:

$$I_z = 2\left[\frac{0.25(1.75)^3}{12} + \left(\frac{1.75}{2} - 1.239\right)^2(0.25)(1.75)\right] + \frac{2(0.25)^3}{12} + (1.875-1.239)^2(0.25)(2)$$

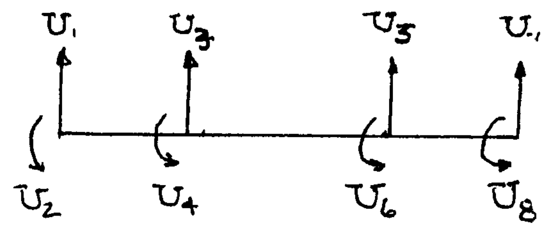
$$I_z = 0.562 \text{ IN}^4$$

ELEMENTS 1 AND 3 $\quad \dfrac{EI_z}{L^3} = 5.62(10^3)$

$$[K^{(1)}] = [K^{(3)}] = \begin{bmatrix} 67.44 & 337.2 & -67.44 & 337.2 \\ & 2248 & -337.2 & 1124 \\ & \text{SYM} & 67.44 & -337.2 \\ & & & 2248 \end{bmatrix}(10^3)$$

ELEMENT 2 $\quad \dfrac{EI_z}{L^3} = 0.703\,(10^3)$

$$[K^{(3)}] = \begin{bmatrix} 8.43 & 42.18 & 8.43 & 42.18 \\ & 1125 & -42.18 & 562.5 \\ & & 8.43 & -42.18 \\ & \text{SYM} & & 1125 \end{bmatrix} (10^3)$$



THE COMPLETE ASSEMBLED STIFFNESS MATRIX IS

$$\begin{bmatrix} 67.44 & 337.2 & -67.44 & 337.2 & 0 & 0 & 0 & 0 \\ & 2248 & -337.2 & 1124 & 0 & 0 & 0 & 0 \\ & & 75.87 & -295 & 8.43 & 42.18 & 0 & 0 \\ & & & 3373 & -42.18 & 562.5 & 0 & 0 \\ & & & & 75.87 & 295 & -67.44 & 337.2 \\ & \text{SYM} & & & & 3373 & -337.2 & 1124 \\ & & & & & & 67.44 & -337.2 \\ & & & & & & & 2248 \end{bmatrix} (10^3)$$

THE NODAL LOAD VECTOR IS

$$\{F\} = \begin{Bmatrix} R_1 \\ 0 \\ -200 \\ 0 \\ -300 \\ 0 \\ R_7 \\ 0 \end{Bmatrix}$$

APPLY THE CONTRAINT CONDITIONS $U_1 = U_7 = 0$ AND SOLVE REMAINING 6×6 SYSTEM TO OBTAIN

$$U_2 = -8.7\,(10^{-4}) \text{ RAD}$$

$$U_3 = -8.47\,(10^{-3}) \text{ IN}$$

$$U_4 = -8\,(10^{-4}) \text{ RAD}$$

$$U_5 = -1.53\,(10^{-2}) \text{ IN.}$$

$$U_6 = 9.7\,(10^{-4}) \text{ RAD}$$

$$U_8 = 1.8\,(10^{-3}) \text{ RAD}$$

4.16

$$U_e = \frac{E}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)\left(\int_A y^2\,dA\right)dx$$

IN THIS CASE

$$I_z = \int_A y^2\,dA$$

IS NOT CONSTANT BUT VARIES WITH AXIAL POSITION SINCE

$$y = h - \frac{h_1}{2L}x = h\left(1 - \frac{x}{2L}\right)$$

SO

$$\int_A y^2\,dA = t\int_{-y(x)}^{y(x)} y^2\,dy = \frac{2t}{3}\,y^3(x)$$

$$= \frac{2t}{3}\,h^3\left(1 - \frac{x}{2L}\right)^3$$

THE STRAIN ENERGY EXPRESSION BECOMES

$$U_e = \frac{Eth^3}{3}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2\left(1 - \frac{x}{2L}\right)^3 dx$$

$$U_e = \frac{Eth^3}{3}\int_0^L \left(\frac{d^2N_1}{dx^2}v_1 + \frac{d^2N_2}{dx^2}\theta_1 + \frac{d^2N_3}{dx^2}v_2 + \frac{d^2N_4}{dx^2}\theta_4\right)^2\left(1 - \frac{x}{2L}\right)^3 dx$$

4.17

$$K_{11} = \frac{2Eth^3}{3}\int_0^L \left(\frac{d^2N_1}{dx^2}\right)^2\left(1-\frac{x}{2L}\right)^3 dx$$

$$K_{11} = \frac{2Eth^3}{3}\int_0^L \left(\frac{6}{L^2}\right)^2\left(2\frac{x}{L}-1\right)^2\left(1-\frac{x}{2L}\right)^3 dx$$

$$K_{11} = \frac{2(36)Eth^3}{3L^4}\int_0^L \left(2\frac{x}{L}-1\right)^2\left(1-\frac{x}{2L}\right)^3 dx$$

LETTING $\xi = \frac{x}{L}$ THIS BECOMES

$$K_{11} = \frac{24\,Eth^3}{L^3}\int_0^1 (2\xi-1)^2\left(1-\frac{\xi}{2}\right)^3 d\xi$$

$$K_{11} = \frac{3Eth^3}{L^3}\int_0^1 (2\xi-1)^2(2-\xi)^3 d\xi$$

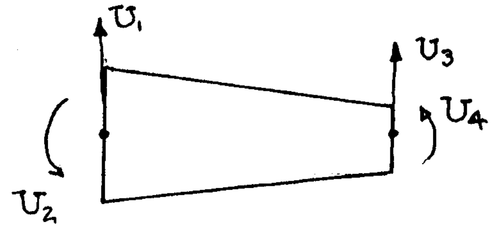
$$K_{11} = \frac{3Eth^3}{L^3}\int_0^1 \left(8-44\epsilon+86\epsilon^2-73\epsilon^3+28\epsilon^4-4\epsilon^5\right)d\epsilon$$

$$K_{11} = \frac{3Eth^3}{L^3}\left(8-22+\frac{86}{3}-\frac{73}{4}+\frac{28}{5}-\frac{4}{6}\right)$$

$$K_{11} = \frac{243\,Eth^3}{60\,L^3}$$

4.18 $\quad h = 0.5$ IN. $\quad t = 0.25$ IN. $\quad L = 12$ IN.

(a) $$\frac{Eth^3}{60L^3} = \frac{10(10^6)(0.25)(0.5)^3}{60(12)^3} = 3.014$$



USING A SINGLE ELEMENT WITH $U_1 = U_2 = 0$ WE HAVE

$$\begin{bmatrix} 732.4 & -3147 \\ -3147 & 19531 \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 0 \end{Bmatrix}$$

YIELDING $\quad U_3 = -4.44(10^{-2})$ IN.

$U_4 = -7.2(10^{-3})$ IN.

(b) ELEMENT 1 $\quad I_z = \frac{0.25(0.875)^3}{12} = 1.40(10^{-2})$ IN^4

$L = 6$ IN.

$$\frac{EI_z}{L^3} = 6.48(10^2)$$

$$[K^{(1)}] = \begin{bmatrix} 77.76 & 233.3 & -77.76 & 233.3 \\ & 933.1 & -233.3 & 466.6 \\ \text{SYM} & & 77.76 & -233.3 \\ & & & 933.1 \end{bmatrix} (10^2)$$

ELEMENT 2 $\quad I_z = \dfrac{0.25(0.625)^3}{12} = 5.1(10^{-3}) \text{ IN}^4$

$L = 6 \text{ IN.} \quad \dfrac{EI_z}{L^3} = 2.35(10^2)$

$$[K^{(2)}] = \begin{bmatrix} 28.2 & 84.6 & -28.2 & 84.6 \\ & 338.4 & -84.6 & 169.2 \\ \text{SYM} & & 28.2 & -84.6 \\ & & & 338.4 \end{bmatrix} (10^2)$$

AGAIN USING THE FIXED SUPPORT CONDITIONS, THE ASSEMBLED EQUATIONS FOR THE NON-CONSTRAINED DISPLACEMENTS ARE:

$$10^2 \begin{bmatrix} 105.8 & -148.7 & -28.2 & 84.6 \\ & 1271.5 & -84.6 & 169.2 \\ \text{SYM} & & 28.2 & -84.6 \\ & & & 338.4 \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -10 \\ 0 \end{Bmatrix}$$

THE TWO STRAIGHT-ELEMENT SOLUTION IS

$$U_3 = -1.30(10^{-2}) \text{ IN.}$$
$$U_4 = -3.89(10^{-3}) \text{ RAD}$$
$$U_5 = -5.05(10^{-2}) \text{ IN.}$$
$$U_6 = -7.4(10^{-3}) \text{ RAD.}$$

(c) THE TAPERED ELEMENT SOLUTION GIVES A LOWER TRANSVERSE DISPLACEMENT (88%) WHILE THE FREE-END SLOPE IS ESSENTIALLY THE SAME FOR (a) AND (b).

(d) STRESSES: VIA EQ. 4.33 EVALUATED AT NODE 1

$$\sigma_{MAX}^{TAPER} = \pm 3250 \text{ psi}$$

$$\sigma_{MAX}^{STRAIGHT} = \pm 3810 \text{ psi}$$

4.19

TO CONVERT THE NODAL EQUILIBRIUM EQUATIONS TO GLOBAL COORDINATES, WE MUST TRANSFORM DISPLACEMENTS AND FORCES FROM THE ELEMENT SYSTEM TO THE GLOBAL SYSTEM. DISPLACEMENT TRANSFORMATION IS PER EQ. 4.63:

$$\{\delta\} = [R]\{U\} \qquad (1)$$

SO WE HAVE

$$[K_e]\{\delta\} = [K_e][R]\{U\} = \{f_e\} \qquad (2)$$

THE RIGHT-HAND SIDE $\{f_e\}$ IS THE NODAL FORCE VECTOR IN THE ELEMENT COORDINATE SYSTEM. EXPRESSING THE NODAL FORCES IN THE GLOBAL SYSTEM, WE HAVE

$$F_1 = F_{1x} = f_{1x}\cos\varphi - f_{1y}\sin\varphi$$

$$F_2 = F_{1y} = f_{1x}\sin\varphi + f_{1y}\cos\varphi$$

$$F_3 = M_1$$

$$F_4 = F_{2x} = f_{2x}\cos\psi - f_{2y}\sin\psi$$

$$F_5 = F_{2y} = f_{2x}\sin\psi + f_{2y}\cos\psi$$

$$F_6 = M_2$$

THE FORCE TRANSFORMATION EQUATIONS ARE, IN MATRIX FORM:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{bmatrix} c\psi & -s\psi & 0 & 0 & 0 & 0 \\ s\psi & c\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c\psi & -s\psi & 0 \\ 0 & 0 & 0 & s\psi & c\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ M_1 \\ f_{2x} \\ f_{2y} \\ M_2 \end{Bmatrix}$$

$$\{F\} = [R]^T\{f\}$$

HENCE, IF WE MULTIPLY ABOVE BY $[R]^T$ WE OBTAIN THE GLOBAL EQUILIBRIUM EQUATIONS

$$[R]^T[k_e][R]\{U\} = [R]^T\{f\} = \{F\}$$

AND $[K_e] = [R]^T[k_e][R]$

4.20 THE ASTUTE STUDENT WILL USE THE MATRIX NOTATION

$$U_e = \frac{1}{2}\{\delta\}^T[k_e]\{\delta\}$$

AND TRANSFORM TO GLOBAL DISPLACEMENTS TO OBTAIN

$$U_e = \frac{1}{2}\{U\}^T[R]^T[k_e][R]\{U\}$$

AND WRITE

$$\Pi_p = U_e - W$$

$$= \frac{1}{2}\{U\}^T[R]^T[k_e][R]\{U\} - \{U\}^T\{F\}$$

$$= \frac{1}{2}\{U\}^T[K]\{U\} - \{U\}^T\{F\}$$

THE PRINCIPLE REQUIRES

$$\frac{\partial \Pi_p}{\partial U_i} = 0 \qquad i = 1,6$$

THE ENERGY CAN BE WRITTEN IN SUMMATION FORM AS:

$$\Pi_p = \frac{1}{2}\sum_{j=1}^{6} K_{1j} U_1 U_j + \frac{1}{2}\sum_{j=1}^{6} K_{2j} U_2 U_j + \frac{1}{2}\sum_{j=1}^{6} K_{3j} U_3 U_j$$

$$+ \frac{1}{2}\sum_{j=1}^{6} K_{4j} U_4 U_j + \frac{1}{2}\sum_{j=1}^{6} K_{5j} U_5 U_j + \frac{1}{2}\sum_{j=1}^{6} K_{6j} U_6 U_j$$

$$- \sum_{i=1}^{6} U_i F_i$$

$$\frac{\partial \Pi_p}{\partial U_1} = K_{11} U_1 + \frac{1}{2} K_{12} U_2 + \frac{1}{2} K_{13} U_3 + \frac{1}{2} K_{14} U_4$$

$$+ \frac{1}{2} K_{15} U_5 + \frac{1}{2} K_{16} U_6 + \frac{1}{2} K_{21} U_2$$

$$+ \frac{1}{2} K_{31} U_3 + \frac{1}{2} K_{41} U_4 + \frac{1}{2} K_{51} U_5$$

$$+ \frac{1}{2} K_{61} U_6 - F_1 = 0$$

AS WE KNOW SYMMETRY EXISTS FOR AN ELASTIC ELEMENT, THE PREVIOUS EQUATION IS

$$K_{11} U_1 + K_{12} U_2 + K_{13} U_3 + K_{14} U_4 + K_{15} U_5$$

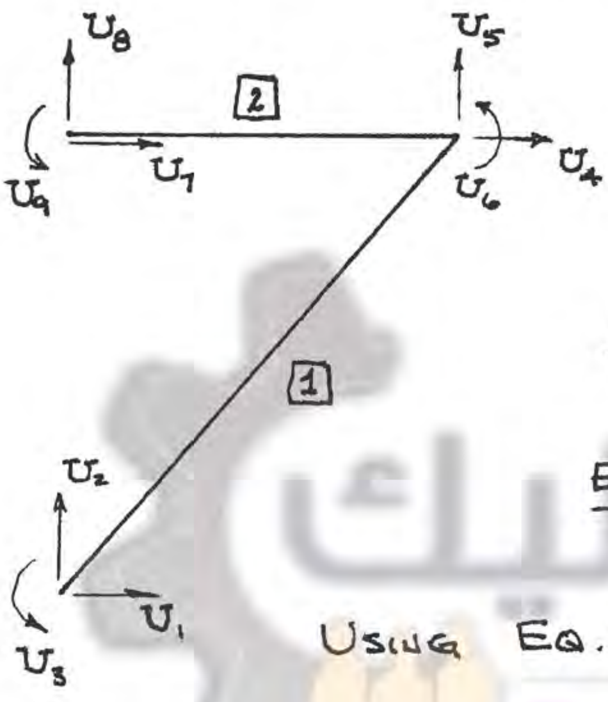
$$+ K_{16} U_6 = F_1$$

THE ADDITIONAL FIVE EQUILIBRIUM EQUATIONS ARE DERIVED BY AN ANALOGOUS PROCEDURE.

4.21 $\quad E = 10(10^6) \qquad I_z = \frac{2(4)^3}{12} = 10.67 \text{ in}^4$

$A = 8 \text{ in}^2$

$L^{(1)} = 42.43 \text{ in.} \qquad L^{(2)} = 30''$



ELEMENT 1

$$\frac{AE}{L} \cong 1.89(10^6)$$

$$\frac{EI_z}{L^3} \cong 1.40(10^3)$$

USING EQ. 4.62

$$[K^{(1)}] = \begin{bmatrix} 1890 & 0 & 0 & -1890 & 0 & 0 \\ & 16.8 & 356.4 & 0 & -16.8 & 356.4 \\ & & 10082 & 0 & 356.4 & 5041 \\ & \text{SYM} & & 1890 & 0 & 0 \\ & & & & 16.8 & -356.4 \\ & & & & & 10082 \end{bmatrix} 10^3$$

CONVERTING TO GLOBAL COORDINATES VIA EQ. 4.65 WITH $\psi = 45°$ WE HAVE

$$[K^{(1)}] = \begin{bmatrix} 953.1 & 936.3 & -252 & -953.1 & -936.3 & -252 \\ & 953.1 & 252 & -936.3 & -953.1 & 252 \\ & & 10082 & -252 & 252 & 5041 \\ & & & 953.1 & 936.3 & 252 \\ & \text{SYM} & & & 953.1 & -252 \\ & & & & & 10082 \end{bmatrix} (10^3)$$

ELEMENT 2

$$\frac{AE}{L} \cong 2.67(10^6) \qquad \frac{EI_z}{L^3} \cong 3.95(10^3) \qquad \psi = 0°$$

$$[K^{(2)}] = \begin{bmatrix} 2670 & 0 & 0 & -2670 & 0 & 0 \\ & 47.4 & 711 & 0 & -47.4 & 711 \\ & & 14220 & 0 & -711 & 7110 \\ & \text{SYM} & & 2670 & 0 & 0 \\ & & & & 47.4 & -711 \\ & & & & & 14220 \end{bmatrix} (10^3)$$

USING THE GLOBAL DISPLACEMENT NOTATION SHOWN ON THE PREVIOUS PAGE, THE GLOBAL STIFFNESS MATRIX IS

$$[K] = 10^3 \begin{bmatrix} 953.1 & 936.3 & -252 & -953.1 & -936.3 & -252 & 0 & 0 & 0 \\ & 953.1 & 252 & -936.3 & -953.1 & 252 & 0 & 0 & 0 \\ & & 10082 & -252 & 252 & 5041 & 0 & 0 & 0 \\ & & & 3623 & 936.3 & 252 & -2670 & 0 & 0 \\ & & & & 998.5 & -963 & 0 & -47.4 & -711 \\ & & & & & 24302 & 0 & 711 & 7110 \\ & \text{SYM} & & & & & 2670 & 0 & 0 \\ & & & & & & & 47.4 & 711 \\ & & & & & & & & 14220 \end{bmatrix} \text{LB/IN}$$

ELEMENT 2 NODAL LOADS EQUIVALENT TO THE DISTRIBUTED LOAD ARE:

$$\{f\} = \begin{Bmatrix} -300 \\ -1500 \\ -300 \\ 1500 \end{Bmatrix}$$

SO THE GLOBAL FORCE VECTOR IS

$$\{F\} = \begin{Bmatrix} R_{1X} \\ R_{1Y} \\ M_1 \\ 0 \\ -1500 \\ 0 \\ R_{3X} \\ R_{3Y}-300 \\ M_3-1500 \end{Bmatrix}$$

FINALLY, APPLICATION OF THE CONSTRAINTS

$$U_1 = U_2 = U_3 = U_7 = U_8 = U_9 = 0$$

YIELDS

$$10^3 \begin{bmatrix} 3623 & 936.3 & 252 \\ & 998.5 & -963 \\ \text{SYM} & & 24302 \end{bmatrix} \begin{Bmatrix} U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1500 \\ 0 \end{Bmatrix}$$

THE DISPLACEMENTS ARE

$$U_4 = 5.5\,(10^{-4}) \text{ IN.}$$

$$U_5 = -2.10\,(10^{-3}) \text{ IN.}$$

$$U_6 = -8.9\,(10^{-5}) \text{ RAD.}$$

NOTE THAT ROTATION θ AT NODE 3 (U_6) IS ESSENTIALLY ZERO AS THE MOMENT DUE TO THE BENDING LOAD CANCELS THE APPLIED MOMENT.

BACK-SUBSTITUTION GIVES THE REACTIONS AS

$$R_{1x} = 1464 \text{ LB.}$$

$$R_{1y} = 1464 \text{ LB}$$

$$M_1 = 219 \text{ IN}\cdot\text{LB}$$

$$R_{3x} = -1464 \text{ LB.}$$

$$R_{3y} = 336 \text{ LB.}$$

$$M_3 = 2361 \text{ IN}\cdot\text{LB}$$

ELEMENT 1 $\quad \sigma_{MAX} = -301.6$ PSI

ELEMENT 2 $\quad \sigma_{MAX} = 344.5$ PSI

4.22 SOLUTION DEPENDS ON HOW THE CONCENTRATED MOMENT AT NODE 2 IS APPLIED.

(a) IF THE MOMENT IS APPLIED TO ELEMENT 2, SOLUTION THE SAME AS FOR PROBLEM 10

(b) IF THE MOMENT IS APPLIED TO ELEMENT 1, THE SOLUTION CHANGES. IN THIS CASE, EACH ELEMENT IS SUBJECTED TO A MOMENT AT NODE 2. IN EITHER CASE, AN ADDITIONAL DEGREE OF FREEDOM BECAUSE THE ROTATIONS AT NODE 2 ARE UNCOUPLED.

DENOTING THE SEPARATE ROTATIONS AT NODE 2 AS $U_6^{(1)}$ AND $U_6^{(2)}$ AND RE-ASSEMBLING THE SYSTEM EQUATIONS GIVES (FOR THE FOUR UNCONSTRAINED DISPLACEMENTS):

$$
10^3 \begin{bmatrix} 3623 & 936.3 & 252 & 0 \\ & 998.5 & -252 & -711 \\ & \text{SYM} & 10082 & 0 \\ & & & 14220 \end{bmatrix} \begin{Bmatrix} U_4 \\ U_5 \\ U_6^{(1)} \\ U_6^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1500 \\ -1500 \\ 1500 \end{Bmatrix}
$$

SOLVING GIVES

$$U_4 = 5.5\,(10^{-4}) \text{ IN.}$$

$$U_5 = -2.07\,(10^{-3}) \text{ IN.}$$

$$U_6^{(1)} = -2.1\,(10^{-4}) \text{ RAD}$$

$$U_6^{(2)} = 2.0\,(10^{-6}) \text{ RAD.}$$

4.23 (a) THE MINIMUM NUMBER OF NODES REQUIRED IS SEVEN – ONE AT EACH JOINT AND ONE AT THE LOAD. THIS WOULD LEAD TO USING EIGHT ELEMENTS. GIVEN THE SYMMETRY HOWEVER ONLY FOUR ELEMENTS ARE REQUIRED (MINIMUM).

(b) IF SYMMETRY IS OBSERVED, THE GLOBAL STIFFNESS MATRIX IS 12 × 12.

(c) AT A AND B, DISPLACEMENTS AND ROTATIONS ARE ZERO.

(d) IF THE SYMMETRY IS OBSERVED, THREE DOF ARE ZERO AT THE FIXED SUPPORT AND TWO ARE ZERO AT THE POINT OF APPLICATION OF W. THUS THE REDUCED MATRIX IS 7×7.

(e) PRIMARILY, OBSERVE STRESS DISCONTINUITIES AT ELEMENT-TO-ELEMENT CONNECTIONS.

<u>4.24</u> (a) AS THERE IS NO SYMMETRY, A MINIMUM OF SIX ELEMENTS IS REQUIRED.

(b) 16 × 16 NOTE THAT THERE IS AN "EXTRA" DEGREE OF FREEDOM AT THE PIN JOINT SINCE THE ELEMENT ROTATIONS ARE UNCOUPLED.

(c) DISPLACEMENTS <u>AND</u> ROTATION AT A ARE ZERO. DISPLACEMENTS AT B ARE ZERO.

(d) 11 × 11

(e) SEE 4.23 (e) ABOVE.

4.25

$$f_{q_{z1}} = \int_0^L q_z N_1(x)\,dx = q_z \int_0^L \left(1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}\right)dx$$

$$= q_z \left(x - \frac{x^3}{L^2} + \frac{x^4}{2L^3} \right|_0^L$$

$$= q_z \left(L - L + \frac{L}{2} \right) = \frac{q_z L}{2}$$

$$M_{q_{z1}} = \int_0^L q_z N_2(x)\,dx = q_z \int_0^L \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2} \right) dx$$

$$= q_z \left(\frac{x^2}{2} - \frac{2x^3}{3L} + \frac{x^4}{4L^2} \right|_0^L$$

$$= q_z \left(\frac{L^2}{2} - \frac{2L^2}{3} + \frac{L^2}{4} \right) = \frac{q_z L^2}{12}$$

$$f_{q_{z2}} = \int_0^L q_z N_3(x)\,dx = q_z \int_0^L \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3} \right) dx$$

$$= q_z \left(\frac{x^3}{L^2} - \frac{x^4}{2L^3} \right|_0^L = \frac{q_z L}{2}$$

$$M_{q_{z2}} = \int_0^L q_z N_4(x)\,dx = q_z \int_0^L \left(\frac{x^3}{L^2} - \frac{x^2}{L} \right) dx$$

$$= q_z \left(\frac{x^4}{4L^2} - \frac{x^3}{3L} \right|_0^L = -\frac{q_z L^2}{12}$$

4.26 $\quad E = 207(10^3)$ MPa

$$I_Y = \frac{60(30)^3}{12} = 1.35(10^5)\ \text{mm}^4$$

$$I_Z = \frac{30(60)^3}{12} = 5.4(10^5)\ \text{mm}^4$$

$$\frac{EI_Y}{L^3} = \frac{207(1.35)(10^8)}{(1.5)^3(10)^9} = 8.28$$

$$\frac{EI_Z}{L^3} = \frac{207(5.4)(10^8)}{(1.5)^3(10^9)} = 33.12$$

XY PLANE (EQ. 4.49)

$$[K]_{XY} = \begin{bmatrix} 397.4 & 2.98(10^5) & -397.4 & 2.98(10^5) \\ & 2.98(10^8) & -2.98(10^5) & 1.49(10^8) \\ & \text{SYM} & 397.4 & -2.98(10^5) \\ & & & 2.98(10^8) \end{bmatrix}$$

XZ PLANE

PER EQ. 4.66

$$[K]_{xz} = \begin{bmatrix} 99.36 & -7.45(10^4) & -99.36 & -7.45(10^4) \\ & 1.12(10^8) & 7.45(10^4) & 0.56(10^8) \\ & & 99.36 & 7.45(10^4) \\ & \text{SYM} & & 1.12(10^8) \end{bmatrix}$$

FOR THE SINGLE ELEMENT

$$\begin{bmatrix} [K]_{xy} & [0] \\ [0] & [K_{xz}] \end{bmatrix} \begin{Bmatrix} V_1 \\ \theta_{z1} \\ V_2 \\ \theta_{z2} \\ W_1 \\ \theta_{y1} \\ W_2 \\ \theta_{y2} \end{Bmatrix} = \begin{Bmatrix} R_{y1} \\ M_{z1} \\ 500 \\ 0 \\ R_{z1} \\ M_{z1} \\ 300 \\ 0 \end{Bmatrix}$$

THE CONSTRAINT CONDITIONS ARE

$$V_1 = \theta_{z1} = W_1 = \theta_{y1} = 0$$

SO THE EQUATIONS FOR THE NON CONSTRAINED DISPLACEMENTS BECOME

$$
\begin{bmatrix} 397.4 & -2.98(10^5) & 0 & 0 \\ -2.98(10^5) & 2.98(10^8) & 0 & 0 \\ 0 & 0 & 99.36 & 7.45(10^4) \\ 0 & 0 & 7.45(10^4) & 1.12(10^8) \end{bmatrix} \begin{Bmatrix} V_2 \\ \theta_{z2} \\ W_2 \\ \theta_{y2} \end{Bmatrix} = \begin{Bmatrix} 500 \\ 0 \\ 300 \\ 0 \end{Bmatrix}
$$

FROM WHICH

$$V_2 = 5.03 \text{ mm}$$

$$\theta_{z2} = 5.03(10^{-3}) \text{ RAD}$$

$$W_2 = 6.02 \text{ MM}$$

$$\theta_{y2} = 4.01(10^{-3}) \text{ RAD}$$

MAXIMUM STRESS ARISING FROM XY-PLANE BENDING IS GIVEN BY EQ. 4.33 AS

$$\sigma = \pm 30(207)(10^3)\left[\frac{6}{1500^2}(5.03) - \frac{2}{1500}(5.03)(10^{-3})\right]$$

$$= \pm 41.6 \text{ MPa}$$

OWING TO THE DIRECTION OF BENDING, THIS STRESS IS COMPRESSIVE AT $y = +30$ MM, TENSILE AT $y = -30$ MM.

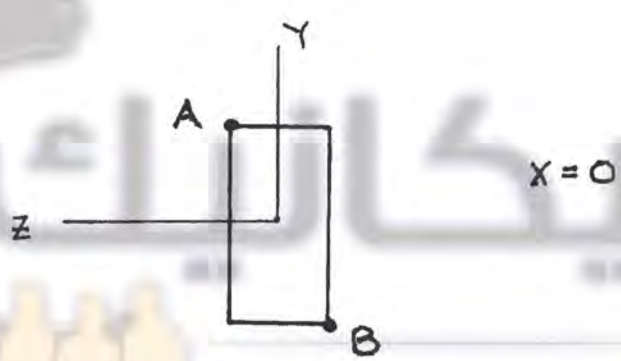
SIMILARLY (USING w_z AND θ_{yz}), MAXIMUM STRESS FROM XZ-PLANE BENDING IS

$$\sigma = \pm 15(207)(10^3)\left[\frac{6}{1500^2}(6.02) - \frac{2}{1500}(4.01)(10^{-3})\right]$$

$$= \pm 33.2 \text{ MPa}$$

COMPRESSIVE AT $z = +15$ MM

TENSILE AT $z = -15$ MM.



MAXIMUM BENDING STRESS IS 74.8 MPa

COMPRESSIVE AT A

TENSILE AT B

4.27 THE STIFFNESS MATRIX IS THE SAME AS IN PROBLEM 4.26. THE LOAD VECTOR BECOMES

$$\{F\} = \begin{Bmatrix} f_{y2} \\ M_{z2} \\ f_{z2} \\ M_{y2} \end{Bmatrix} = \begin{Bmatrix} q_y \frac{L}{2} \\ -\frac{q_y L^2}{12} \\ \frac{q_z L}{2} \\ -\frac{q_z L^2}{12} \end{Bmatrix} = \begin{Bmatrix} 450 \\ -1.125(10^5) \\ 300 \\ -7.5(10^4) \end{Bmatrix}$$

(MOMENT EXPRESSED IN N·mm.)

FOR THIS LOADING CONDITION, THE DISPLACEMENT SOLUTION IS

$$v_2 = 3.40 \text{ mm}$$

$$\theta_{z2} = 3.02(10^{-3}) \text{ RAD}$$

$$w_2 = 5.02 \text{ mm}$$

$$\theta_{y2} = 2.67(10^{-3}) \text{ RAD}$$

CHAPTER 6

6.1 FOR RIGID BODY TRANSLATION $V_1 = V_2$ AND $\theta_1 = \theta_2 = 0$

$$v(x) = (N_1 + N_3)v_1$$

$$= \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)v_1 = v_1$$

6.2 FOR RIGID BODY ROTATION $\theta_1 = \theta_2 = \theta$ AND $v_1 = -\frac{L}{2}\theta \quad v_2 = \frac{L}{2}\theta$

$$v(x) = (N_3 - N_1)\frac{L}{2}\theta + (N_2 + N_4)\theta$$

$$= \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3} - 1 + \frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)\frac{L}{2}\theta$$

$$+ \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2} - \frac{x^2}{L} + \frac{x^3}{L^2}\right)\theta$$

$$= \left(\frac{3x^2}{L} - \frac{2x^3}{L^2} - \frac{L}{2}\theta\right) - \left(x - \frac{3x^2}{L} + \frac{2x^3}{L^2}\right)\theta$$

$$= -\frac{L}{2}\theta + x\theta = v_1 + x\theta$$

$$\frac{dv}{dx} = \theta = \text{CONSTANT}$$

6.3

$$\frac{d^2v}{dx^2} = \frac{d^2N_1}{dx^2}v_1 + \frac{d^2N_2}{dx^2}\theta_1 + \frac{d^2N_3}{dx^2}v_2 + \frac{d^2N_4}{dx^2}\theta_2$$

$$= \left(\frac{12x}{L^3} - \frac{6}{L^2}\right)v_1 + \left(\frac{6x}{L^2} - \frac{4}{L}\right)\theta_1$$

$$+ \left(\frac{6}{L^2} - \frac{12x}{L^3}\right)v_2 + \left(\frac{6x}{L^2} - \frac{2}{L}\right)\theta_2$$

IN PURE BENDING, $v_1 = v_2$, $\theta_1 = -\theta_2$ SO

$$\frac{d^2v}{dx^2} = \frac{2}{L}\theta_2 = \text{CONSTANT}$$

6.4

IF SHEAR FORCE IS CONSTANT, $\frac{d^3v}{dx^3}$ IS CONSTANT. DIFFERENTIATING AGAIN IN PROBLEM 6.3

$$\frac{d^3v}{dx^3} = \frac{12}{L^3}(v_1 - v_2) + \frac{6}{L^2}(\theta_1 + \theta_2)$$

$$= \text{CONSTANT}$$

6.5

$$N_1(s) = \frac{(s-s_2)(s-s_3)(s-s_4)}{(s_1-s_2)(s_1-s_3)(s_1-s_4)}$$

$$s_1 = 0 \quad s_2 = \frac{1}{3} \quad s_3 = \frac{2}{3} \quad s_4 = 1$$

$$N_1(s) = \frac{(s-\frac{1}{3})(s-\frac{2}{3})(s-1)}{(-\frac{1}{3})(-\frac{2}{3})(-1)}$$

$$N_1(s) = -\frac{9}{2}(s-\tfrac{1}{3})(s-\tfrac{2}{3})(s-1)$$

6.6

$$L_2(s) = \frac{(s-s_1)(s-s_3)(s-s_4)}{(s_2-s_1)(s_2-s_3)(s_2-s_4)}$$

$$L_3(s) = \frac{(s-s_1)(s-s_2)(s-s_4)}{(s_3-s_1)(s_3-s_2)(s_3-s_4)}$$

6.7

$$N_1(x) = L_1(x) = \frac{(x-2.25)(x-2.5)(x-2.75)(x-3)}{(-0.25)(-0.5)(-0.75)(-1)}$$

$$N_2(x) = L_2(x) = \frac{(x-2)(x-2.5)(x-2.75)(x-3)}{0.25(-0.25)(-0.5)(-0.75)}$$

$$N_3(x) = L_3(x) = \frac{(x-2)(x-2.25)(x-2.75)(x-3)}{0.5(0.25)(-0.25)(-0.5)}$$

$$N_4(x) = L_4(x) = \frac{(x-2)(x-2.25)(x-2.5)(x-3)}{0.75(0.5)(0.25)(-0.25)}$$

$$N_5(x) = L_5(x) = \frac{(x-2)(x-2.25)(x-2.5)(x-2.75)}{(1)(0.75)(0.5)(0.25)}$$

6.8 THE POLYNOMIAL

$$P(x,y) = a_0 + a_1 x^2 + a_2 xy + a_3 y^2$$

IS NOT SUITABLE FOR THE ELEMENT DESCRIBED BECAUSE THE FIRST PARTIAL DERIVATIVES

$$\frac{\partial P}{\partial x} = 2a_1 x + a_2 y$$

AND

$$\frac{\partial P}{\partial Y} = a_2 X + 2a_3 Y$$

CANNOT ASSUME CONSTANT VALUES. THUS THE COMPLETENESS REQUIRMENT IS NOT SATISFIED.

6.9

(1) $P(x,y) = a_0 + a_1 x + a_2 xy + a_3 y + a_4 x^2$

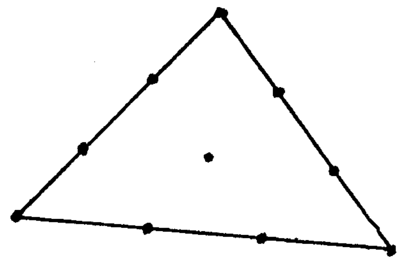
$$+ a\, y^2 + a\, x^2 y + a\, xy^2 + a\, x^2 y^2$$

THIS FORM IS SUITABLE FOR THE 9-NODE TRIANGLE AND COULD BE USED WITH A 9-NODE RECTANGLE HAVING ONE INTERIOR NODE.

(2) DROP THE x^2y^2 TERM AND ADD

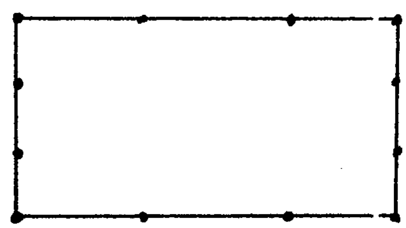
$$a_8 x^3 y + a_9 x y^3$$

NOT A VERY USEFUL FORM BUT COULD BE USED FOR A 10-NODE TRIANGLE HAVING ONE INTERIOR NODE AS SHOWN BELOW.



(3) ADD $a_{10}X^4 + a_{11}Y^4$ (STILL OMITTING X^2Y^2)

THIS FORM COULD BE APPLIED TO A 12-NODE RECTANGLE AS SHOWN



STUDENTS WILL FIND OTHER FORMS BUT MOST OF THOSE ARE NOT USEFUL.

<u>6.10</u> THESE POLYNOMIALS MUST CONTAIN ALL THREE LINEAR TERMS, THAT IS

a_0, a_1X, a_2Y, a_3Z: SO LET

$$P_0(X,Y,Z) = a_0 + a_1X + a_2Y + a_3Z$$

TO $P_0(x,y)$ CAN BE ADDED:

(1) ALL THE QUADRATIC TERMS

(2) xy, xz, yz

(3) x^2, y^2, z^2

AND SYMMETRY IS MAINTAINED.

TO THE THREE COMBINATIONS ABOVE, ONE CAN ADD THE CUBIC TERMS

(4) xyz

(5) x^3, y^3, z^3

(6) $x^2y, xy^2, y^2z, yz^2, xz^2, x^2z$

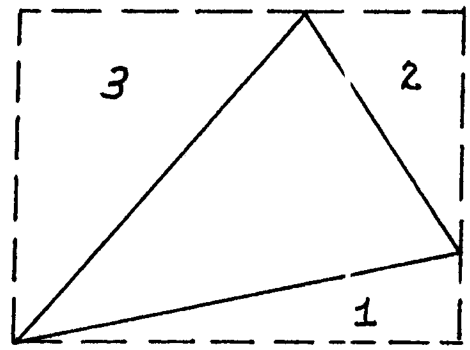
(7) (4) AND (5)

(8) (4) AND (6)

(9) (5) AND (6)

THUS THERE ARE 27 SUCH POLYNOMIALS THAT ARE INCOMPLETE BUT SYMMETRIC.

6.11 "SURROUND" THE TRIANGLE WITH A RECTANGLE CONTAINING THE NODES AS SHOWN BELOW



AREA OF THE TRIANGULAR ELEMENT IS THEN

$$A = A_{RECT} - A_1 - A_2 - A_3$$

$$A = (X_2 - X_1)(Y_3 - Y_1) - \tfrac{1}{2}(X_2 - X_1)(Y_2 - Y_1)$$

$$-\tfrac{1}{2}(X_2 - X_3)(Y_3 - Y_2) - \tfrac{1}{2}(X_3 - X_1)(Y_3 - Y_1)$$

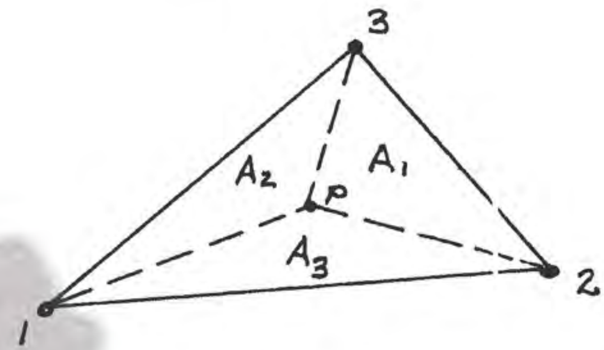
EXPANDING AND SIMPLIFYING

$$A = \tfrac{1}{2}(X_2Y_3 - Y_2X_3) - \tfrac{1}{2}X_1(Y_3 - Y_2) + \tfrac{1}{2}Y_1(X_3 - X_2)$$

$$= \frac{1}{2}\begin{vmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{vmatrix}$$

6.12 WE NEED TO SHOW THAT

$$L_1 = N_1$$
$$L_2 = N_2$$
$$L_3 = N_3$$



P ⇒ (X,Y) ⇒ ANY INTERIOR POINT

USING EQ. 6.38

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & X & Y \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{vmatrix}$$

$$A_1 = \frac{1}{2}\left[(X_2Y_3 - X_3Y_2) + (Y_2 - Y_3)X + (X_3 - X_2)Y \right]$$

$L_1 = \dfrac{A_1}{A}$ IS IDENTICAL TO THE FIRST OF EQS. 6.37.

PROOF FOR L_2 AND L_3 IS ANALOGOUS.

6.13

$$N_1 = \frac{9}{2} L_1 \left(L_1 - \frac{1}{3}\right)\left(L_1 - \frac{2}{3}\right)$$

$$N_2 = \frac{9}{2} L_2 \left(L_2 - \frac{1}{3}\right)\left(L_2 - \frac{2}{3}\right)$$

$$N_3 = \frac{9}{2} L_3 \left(L_3 - \frac{1}{3}\right)\left(L_3 - \frac{2}{3}\right)$$

$$N_4 = \frac{27}{2} L_1 L_2 \left(L_1 - \frac{1}{3}\right)$$

$$N_5 = \frac{27}{2} L_1 L_2 \left(L_2 - \frac{1}{3}\right)$$

$$N_6 = \frac{27}{2} L_2 L_3 \left(L_2 - \frac{1}{3}\right)$$

$$N_7 = \frac{27}{2} L_2 L_3 \left(L_3 - \frac{1}{3}\right)$$

$$N_8 = \frac{27}{2} L_1 L_3 \left(L_3 - \frac{1}{3}\right)$$

$$N_9 = \frac{27}{2} L_1 L_3 \left(L_1 - \frac{1}{3}\right)$$

$$N_{10} = 27 L_1 L_2 L_3$$

6.14 AT NODE 10, $L_1 = L_2 = L_3 = \frac{1}{3}$. THUS TOTAL AREA $A = A_1 + A_2 + A_3$ IS UNIFORMLY DISTRIBUTED AROUND NODE 10. A MATHEMATICAL PROOF IS ALGEBRAICALLY TEDIOUS BUT NOT NECESSARY.

6.15

(a) $$\iint_A L_2^2 (2L_2 - 1)^2 dA = \iint_A (4L_2^4 - 4L_2^3 + L^2) dA$$

$$= \iint_A 4L_2^4 dA - \iint_A 4L_2^3 dA + \iint_A L^2 dA$$

$$= 4\left[\frac{4!}{6!} - \frac{3!}{5!}\right](2A) + \left(\frac{2!}{4!}\right)(2A)$$

$$= 8A\left[\frac{1}{30} - \frac{1}{20}\right] + 2A\left(\frac{1}{12}\right) = \frac{A}{30}$$

(b) $$\iint_A 4L_1L_2L_3(2L_1 - 1) dA$$

$$= 4\iint_A 2L_1^2 L_2 L_3 dA - 4\iint_A L_1L_2L_3 dA$$

$$= 16A\left(\frac{2!1!1!}{7!}\right) - 8A\left(\frac{1}{5!}\right)$$

$$= 16A\left(\frac{2}{5040}\right) - 8A\left(\frac{1}{120}\right) = -0.0603A$$

(c) $\iint_A L_1^3 L_2 L_3 \, dA = 2A\left(\frac{3!1!1!}{7!}\right) = 0.00238A$

(d) SAME AS (c)

<u>6.16</u> $\sum_{i=1}^{4} N_i(r,s) = 1$

FROM EQ. 6.56 IS BASED ON USE OF THE FIELD VARIABLE VALUE AT EACH NODE. (THAT IS, DERIVATIVES OF THE FIELD VARIABLE ARE <u>NOT</u> NODAL VARIABLES.) TO SATISFY THE COMPLETENESS REQUIREMENT, THE FIELD VARIABLE MUST BE CAPABLE OF ASSUMING A CONSTANT VALUE. IF $\phi = C$ = CONSTANT THEN

$$\phi = N_1 C + N_2 C + N_3 C + N_4 C$$

$$= (N_1 + N_2 + N_3 + N_4) C$$

$$\therefore \quad \Sigma N_i = 1$$

6.17

$$\frac{\partial \phi}{\partial r} = \frac{1}{4}\left[(s-1)\phi_1 + (1-s)\phi_2 + (1+s)\phi_3 - (1+s)\phi_4\right]$$

$$\frac{\partial \phi}{\partial r} = \frac{1}{4}\left[(1-s)(\phi_2-\phi_1) + (1+s)(\phi_3-\phi_4)\right]$$

IF $\frac{\partial \phi}{\partial r}$ = CONSTANT, THEN $\phi_2-\phi_1 = \phi_3-\phi_4 = \Delta\phi$ AND THE EXPRESSION BECOMES

$$\frac{\partial \phi}{\partial r} = \frac{\Delta \phi}{4} = \text{CONSTANT}$$

PROOF FOR $\frac{\partial \phi}{\partial s}$ IS SIMILAR.

6.18 ALONG 2-3, THE INTERPOLATION FUNCTIONS ASSOCIATED WITH NODES 1 AND 4 EVALUATE TO ZERO. ALSO ALONG 2-3 THE INTEROLATION FUNCTIONS FOR EACH ELEMENT VARY LINEARLY FROM ZERO TO ONE (OR ONE TO ZERO). HENCE ONLY NODAL VALUES AT NODES 2 AND 3 AFFECT THE VALUES ALONG 2-3 ⇒ THUS CONTINUITY. (THE STUDENT CAN WRITE THE EQUATIONS FOR EACH ELEMENT TO ARRIVE AT THE SAME RESULT.)

6.19 LET 1-2-3-4 BE ELEMENT 1 AND 2-5-6-3 BE ELEMENT 2.

FOR ELEMENT 1 ALONG 2-3 $r=1$ SO

$$\phi_{2-3}^{(1)} = \tfrac{1}{2}(1-s)\phi_2 + \tfrac{1}{2}(1+s)\phi_3$$

FOR ELEMENT 2 ALONG 2-3 $r=-1$ SO

$$\phi_{2-3}^{(2)} = \tfrac{1}{2}(1-s)\phi_2 + \tfrac{1}{2}(1+s)\phi_3$$

$\therefore$ CONTINUITY EXISTS ACROSS THE BOUNDARY

6.20 ϕ WILL BE CONTINUOUS ALONG AND ACROSS THE BOUNDARY AS IN P6.19. ALONG 2-5, $s=-1$ SO THE PARTIAL DERIVATIVES ARE

$$\frac{\partial\phi}{\partial r} = -\frac{1}{2}\phi_2 + \frac{1}{2}\phi_5 = \frac{1}{2}(\phi_5 - \phi_2)$$

$$\frac{\partial\phi}{\partial s} = \frac{1}{4}(r-1)\phi_2 - \frac{1}{4}(r+1)\phi_5 + \frac{1}{4}(1+r)\phi_6 + \frac{1}{4}(r-1)\phi_3$$

BOTH PARTIAL DERIVATIVES ARE CONTINUOUS ALONG 2-5 BUT ONLY $\partial\phi/\partial r$ IS CONTINUOUS

ACROSS THE BOUNDARY (WHEN, OF COURSE, THIS IS A BOUNDARY BETWEEN ELEMENTS). CONTINUITY EXISTS ONLY IF THE FIELD VARIABLE OR ANY DERIVATIVE DEPENDS ONLY ON VALUES AT NODES LOCATED ON THE BOUNDARY.

6.21 LET ELEMENT 1 = 1-2-3-7-11-10-9-6
ELEMENT 2 = 3-4-5-8-13-12-11-7

IN THE ELEMENT COORDINATES:

FOR ELEMENT 1, 3-7-11 CORRESPOND TO $r=1$ AND NODES 2-6-3

FOR ELEMENT 2, 3-7-11 CORRESPOND TO $r=-1$ AND NODES 1-8-4.

THUS

$$\phi^{(1)}_{3\text{-}7\text{-}11} = -\frac{1}{2}s(1-s)\phi_3 + (1-s^2)\phi_7 + \frac{1}{2}s(1+s)\phi_{11}$$

$$\phi^{(2)}_{3\text{-}7\text{-}11} = -\frac{1}{2}s(1-s)\phi_3 + (1-s^2)\phi_7 + \frac{1}{2}s(1+s)\phi_{11}$$

SINCE THE EXPRESSIONS ARE IDENTICAL, FIELD VARIABLE CONTINUITY IS ASSURED. THE PARTIAL DERIVATIVES ARE

$$\frac{\partial N_1}{\partial r} = \frac{1}{4}(r-1)(1-s) + \frac{1}{4}(1-s)(r+s+1)$$

$$\frac{\partial N_2}{\partial r} = \frac{1}{4}(s-1)(s-r+1) - \frac{1}{4}(r+1)(s-1)$$

$$\frac{\partial N_3}{\partial r} = \frac{1}{4}(1+r)(1+s) + \frac{1}{4}(1+s)(r+s-1)$$

$$\frac{\partial N_4}{\partial r} = \frac{1}{4}(r-1)(1+s) + \frac{1}{4}(1+s)(r-s+1)$$

$$\frac{\partial N_5}{\partial r} = -r(1-s)$$

$$\frac{\partial N_6}{\partial r} = \frac{1}{2}(1-s^2)$$

$$\frac{\partial N_7}{\partial r} = -r(1+s)$$

$$\frac{\partial N_8}{\partial r} = \frac{1}{2}(s^2-1)$$

$$\frac{\partial N_1}{\partial s} = \frac{1}{4}(r-1)(1-s) - \frac{1}{4}(r-1)(r+s+1)$$

$$\frac{\partial N_2}{\partial s} = \frac{1}{4}(r+1)(s-r+1) + \frac{1}{4}(r+1)(s-1)$$

$$\frac{\partial N_3}{\partial s} = \frac{1}{4}(1+r)(r+s-1) + \frac{1}{4}(1+r)(1+s)$$

$$\frac{\partial N_4}{\partial s} = \frac{1}{4}(r-1)(r-s+1) - \frac{1}{4}(r-1)(1+s)$$

$$\frac{\partial N_5}{\partial s} = \frac{1}{2}(r^2-1)$$

$$\frac{\partial N_6}{\partial s} = -s(1+r)$$

$$\frac{\partial N_7}{\partial s} = \frac{1}{2}(1-r^2)$$

$$\frac{\partial N_8}{\partial s} = -s(1-r)$$

FOR ELEMENT 1 ALONG EDGE $r=1$

$$\frac{\partial \phi}{\partial r} = \frac{1}{4}(1-s)(2+s)\phi_1 + \frac{1}{4}(s-1)(2+s)\phi_3$$

$$+ \frac{1}{4}(1+s)(2+s)\phi_{11} + \frac{1}{4}(1+s)(2-s)\phi_9$$

$$+ (s-1)\phi_2 + \frac{1}{2}(1-s^2)\phi_7 - (1+s)\phi_{10} + \frac{1}{2}(s^2-1)\phi_6$$

FOR ELEMENT 2 ALONG EDGE $r=-1$

$$\frac{\partial \phi}{\partial r} = \frac{1}{4}(1-s)(s-2)\phi_3 + \frac{1}{4}(s-1)(s+2)\phi_5$$

$$+\frac{1}{4}(1+s)(s-2)\phi_{13} - \frac{1}{4}(1+s)(2+s)\phi_{11}$$

$$+(1-s)\phi_4 + \frac{1}{2}(1-s^2)\phi_8 + (1+s)\phi_{12} + \frac{1}{2}(s^2-1)\phi_7$$

CLEARLY

$$\left.\frac{\partial \phi}{\partial r}\right|_{\text{ELEMENT 1}} \neq \left.\frac{\partial \phi}{\partial r}\right|_{\text{ELEMENT 2}}$$

SO DERIVATIVE CONTINUITY DOES NOT EXIST.
THE RESULT FOR $\frac{\partial \phi}{\partial s}$ IS SIMILARLY OBTAINED.

6.22

$$\sum_{i=1}^{8} N_i = \frac{1}{4}(r-1)(1-s)(r+s+1) + \frac{1}{4}(r+1)(s-1)(s-r+1)$$

$$+\frac{1}{4}(1+r)(1+s)(r+s-1) + \frac{1}{4}(r-1)(1+s)(r-s+1)$$

$$+\frac{1}{2}(1-r^2)(1-s) + \frac{1}{2}(1+r)(1-s^2) + \frac{1}{2}(1-r^2)(1+s)$$

$$+ \frac{1}{2}(1-r)(1-s^2)$$

$$\Sigma N_i = \frac{1}{4}(r-1)\left[(1-s)(r+s+1)+(1+s)(r-s+1)\right]$$

$$+\frac{1}{4}(r+1)\left[(s-1)(s-r+1)+(1+s)(r+s-1)\right]$$

$$+\frac{1}{2}(1-r^2)(2)+\frac{1}{2}(1-s^2)(2)$$

$$=\frac{1}{4}(r-1)(2)(1+r-s^2)+\frac{1}{4}(r+1)(2)(s^2+r-1)$$

$$+1-r^2+1-s^2$$

$$= r^2+s^2-1+1-r^2+1-s^2 \equiv 1$$

<u>6.23</u> SINCE $L_1+L_2+L_3+L_4=1$

$$x = L_1x_1+L_2x_2+L_3x_3+(1-L_1-L_2-L_3)x_4$$

$$y = L_1y_1+L_2y_2+L_3y_3+(1-L_1-L_2-L_3)y_4$$

$$z = L_1z_1+L_2z_2+L_3z_3+(1-L_1-L_2-L_3)z_4$$

$$[J]=\begin{bmatrix}\frac{\partial x}{\partial L_1} & \frac{\partial y}{\partial L_1} & \frac{\partial z}{\partial L_1}\\ \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} & \frac{\partial z}{\partial L_2}\\ \frac{\partial x}{\partial L_3} & \frac{\partial y}{\partial L_3} & \frac{\partial z}{\partial L_3}\end{bmatrix}=\begin{bmatrix}x_1-x_4 & y_1-y_4 & z_1-z_4\\ x_2-x_4 & y_2-y_4 & z_2-z_4\\ x_3-x_4 & y_3-y_4 & z_3-z_4\end{bmatrix}$$

$$dV = dx\,dy\,dz = |J|\, dL_1\, dL_2\, dL_3$$

$$\therefore\ V = \int_0^1 \int_0^{1-L_3} \int_0^{1-L_2-L_3} |J|\, dL_1\, dL_2\, dL_3$$

SINCE $|J|$ IS CONSTANT

$$V = |J| \int_0^1 \int_0^{1-L_3} \int_0^{1-L_2-L_3} dL_1\, dL_2\, dL_3$$

$$V = |J| \frac{0!\,0!\,0!}{3!} = \frac{|J|}{6}$$

LET $x_{14} = x_1 - x_4$, ETC SO THAT

$$|J| = \begin{vmatrix} x_{14} & y_{14} & z_{14} \\ x_{24} & y_{24} & z_{24} \\ x_{34} & y_{34} & z_{34} \end{vmatrix}$$

$$= x_{14}(y_{24} z_{34} - y_{34} z_{24})$$

$$- y_{14}(x_{24} z_{34} - x_{34} z_{24}) + z_{14}(x_{24} y_{34} - x_{34} y_{24})$$

WHICH EXPANDS TO

$$
\begin{aligned}
|J| = {} & x_1(y_2 z_3 - y_3 z_2 + y_3 z_4 - y_4 z_3 + y_4 z_2 - y_2 z_4) \\
& + y_1(x_3 z_2 - x_2 z_3 + x_2 z_4 - z_4 x_2 + x_4 z_3 - x_3 z_4) \\
& + z_1(x_2 y_3 - x_3 y_2 + x_3 y_4 - x_4 y_3 + x_4 y_2 - x_2 y_4) \\
& + x_4(y_3 z_2 - y_2 z_3) + x_3(y_2 z_4 - y_4 z_2) \\
& + x_2(y_4 z_3 - y_3 z_4)
\end{aligned}
$$

EXPANDING THE DETERMINANT

$$
\begin{vmatrix}
1 & x_1 & y_1 & z_1 \\
1 & x_2 & y_2 & z_2 \\
1 & x_3 & y_3 & z_3 \\
1 & x_4 & y_4 & z_3
\end{vmatrix}
$$

GIVES THE SAME RESULT.

6.24

$$r = L_1 r_1 + L_2 r_2 + L_3 r_3$$

$$= \frac{A_1}{A} r_1 + \frac{A_2}{A} r_2 + \frac{A_3}{A} r_3$$

$$= \frac{1}{A}\left(A_1 r_1 + A_2 r_2 + A_3 r_3\right)$$

PER EQ. 6.38 (REVISED TO CYLINDRICAL COORDINATES)

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \\ 1 & r & z \end{vmatrix}$$

$$A_2 = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$A_3 = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r & z \end{vmatrix}$$

EXPANDING THE DETERMINANTS YIELDS

$$A_1 = r_3 z - r z_3 - (r_2)(z - z_3) + z_2(r - r_3)$$

$$A_2 = r z_3 - z r_3 - r_1(z_3 - z) + z_1(r_3 - r)$$

$$A_3 = r_2 z - z_2 r - r_1(z - z_2) + z_1(r - r_2)$$

$$r_1 A_1 = \cancel{r_1 r_3 z} - r_1 z_3 r - \cancel{r_1 r_2 z} + \cancel{r_1 r_2 z_3} + r_1 z_2 r - \cancel{r_1 r_3 z_2}$$

$$r_2 A_2 = r_2 z_3 r - \cancel{r_2 r_3 z} - \cancel{r_1 r_2 z_3} + \cancel{r_1 r_2 z} + \cancel{r_2 r_3 z_1} - r_2 z_1 r$$

$$r_3 A_3 = \cancel{r_2 r_3 z} - r_3 z_2 r - \cancel{r_1 r_3 z} + \cancel{r_1 r_3 z_2} + r_3 z_1 r - \cancel{r_2 r_3 z_1}$$

$$\Sigma r_i A_i = r(r_1 z_2 - r_2 z_1) - r(r_1 z_3 - r_3 z_1) + r(r_2 z_3 - r_3 z_2) = \frac{1}{2A} r \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$\therefore \quad r_1 A_1 + r_2 A_2 + r_3 A_3 = r$$

6.25 FROM EXAMPLE 6.6

$$I = 2\pi \iint_A L_2(2L_2-1)(4L_1L_2)\, r\, dA$$

$$= 2\pi \iint_A L_2(2L_2-1)(4L_1L_2)(r_1L_1 + r_2L_2 + r_3L_3)\, dA$$

$$= 2\pi \iint_A (2L_2^2 - L_2)(4L_1L_2)(r_1L_1 + r_2L_2 + r_3L_3)\, dA$$

$$= 2\pi \iint_A (8L_1L_2^3 - 4L_1L_2^2)(r_1L_1 + r_2L_2 + r_3L_3)\, dA$$

$$= 2\pi \iint_A \big(8r_1L_1^2L_2^3 - 4r\, L_1^2L_2^2 + 8r_2L_1L_2^4 - 4r_2L_1L_2^3 + 8r_3L_1L_2^3L_3 - 4r_3L_1L_2^2L_3\big)\, dA$$

$$= \Big(16\pi r_1 \frac{2!\,3!}{7!} - 8\pi r_1 \frac{2!\,2!}{!} + 16\pi r_2 \frac{4!}{7!} - 8\pi r_2 \frac{3!}{6!} + 16\pi r_3 \frac{3!}{7!} - 8\pi r_3 \frac{2!}{6!}\Big) 2A$$

$$= \frac{\pi A}{315}\left(-4r_1 + 6r_2 - 2r_3\right)$$

6.26 NOTE: MULTIPLIER 2A IS MISSING IN LAST INTEGRAL. EQUATION SHOULD BE

$$A = \iint dA = \iint dx\,dy = 2A\int_0^1 \int_0^{1-L_2} dL_1\, dL_2$$

$$\iint dA = \iint L_1^0 L_2^0 L_3^0 \, dA = \frac{0!\,0!\,0!}{2!}(2A) = A$$

$$2A\int_0^1 \int_0^{1-L_2} dL_1\, dL_2 = 2A\int_0^1 (1-L_2)\, dL_2$$

$$= 2A\left(L_2 - \frac{L_2^2}{2}\right)\Big|_0^1 = A$$

THE LATTER FORM CORRESPONDS TO INTEGRATING WITH $L_3 = 0$ SINCE L_1, L_2 AND L_3 ARE NOT INDEPENDENT.

6.27

$$x = \frac{1}{4}(1-r)(1-s)x_1 + \frac{1}{4}(1+r)(1-s)x_2 + \frac{1}{4}(1+r)(1+s)x_3 + \frac{1}{4}(1-r)(1+s)x_4$$

$$y = \frac{1}{4}(1-r)(1-s)y_1 + \frac{1}{4}(1+r)(1-s)y_2 + \frac{1}{4}(1+r)(1+s)y_3 + \frac{1}{4}(1-r)(1+s)x_4$$

$r = \frac{1}{2} \qquad s = 0$

$$x = \frac{x_1}{8} + \frac{3x_2}{8} + \frac{3x_3}{8} + \frac{x_4}{8} = 2.3375$$

$$y = \frac{y_1}{8} + \frac{3y_2}{8} + \frac{3y_3}{8} + \frac{y_4}{8} = 2.325$$

6.28 $\qquad r = 0$

$$x = \frac{1}{4}(1-s)(x_1 + x_2) + \frac{1}{4}(1+s)(x_3 + x_4)$$

$$y = \frac{1}{4}(1-s)(y_1 + y_2) + \frac{1}{4}(1+s)(y_3 + y_4)$$

$$x = \frac{1}{4}(8.9 - 0.1s)$$

$$y = \frac{1}{4}(9.2 + s)$$

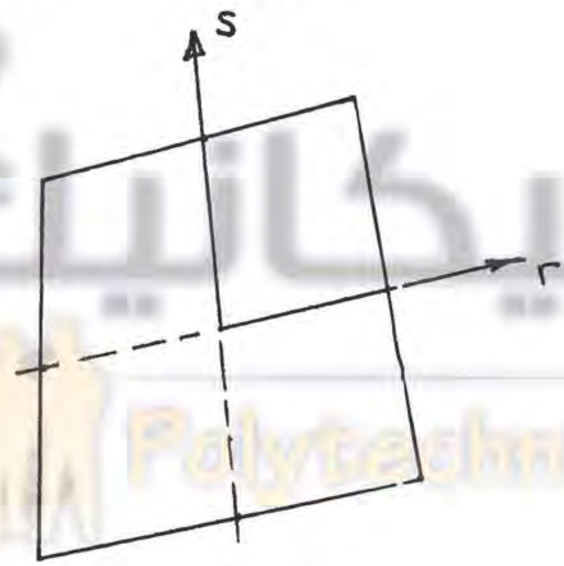
6.29 $\quad s = 0$

$$x = \tfrac{1}{4}(1-r)(x_1 + x_4) + \tfrac{1}{4}(1+r)(x_2 + x_3)$$

$$y = \tfrac{1}{4}(1-r)(y_1 + y_4) + \tfrac{1}{2}(1+r)(y_2 + y_3)$$

$$x = \tfrac{1}{4}(8.9 + 0.9r)$$

$$y = \tfrac{1}{4}(9.2 + 0.2r)$$



6.30 $\quad x = (1-r)x_1 + r x_2$

IN THIS CASE THE MAPPING ACCOMPLISHES NOTHING. THE PARENT ELEMENT IS MAPPED TO ITSELF. THE JACOBIAN "MATRIX" IS THE SCALAR

$$\frac{\partial x}{\partial r} = x_2 - x_1$$

6.31

$$x = (2r-1)(r-1)x_1 + 4r(1-r)x_2 + r(2r-1)x_3$$

(a) THE RELATION BETWEEN PHYSICAL COORDINATES AND PARENT COORDINATES IS QUADRATIC.

(b) GEOMETRY DOES NOT CHANGE SINCE WE STILL HAVE A 3-NODE LINE ELEMENT AFTER MAPPING.

(c) AGAIN [J] IS A SCALAR

$$J = \frac{\partial x}{\partial r} = (4r-3)x_1 + (4-8r)x_2 + (4r-1)x_3$$

(d) $J^{-1} = \frac{1}{J}$ = SCALAR

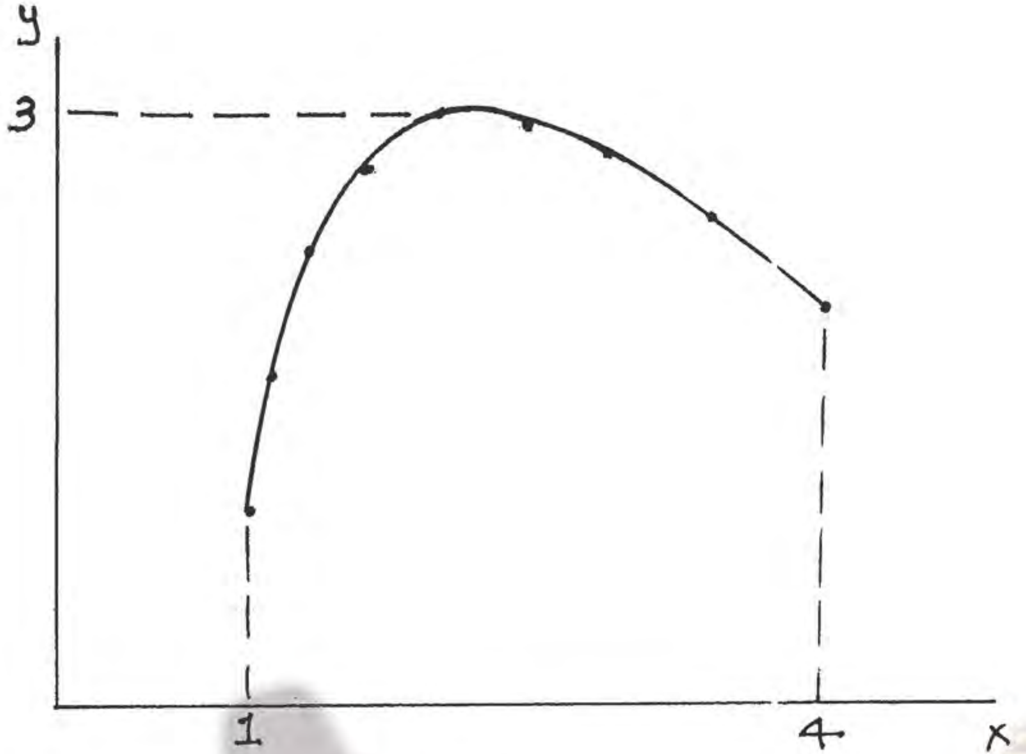
6.32

(a) $x = (2r-1)(r-1) + 4r(1-r)(2) + r(2r-1)(4)$

$$y = (2r-1)(r-1) + 4r(1-r)(3) + r(2r-1)(2)$$

$$x = 2r^2 + r + 1$$

$$y = -6r^2 + 7r + 1$$



(b) THE ELEMENT COULD BE USED IN A CONDUCTION ANALYSIS BUT THE LIKELIHOOD OF FITTING THE ELEMENT TO A PHYSICAL GEOMETRY IS VERY LOW.

6.33

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} r^2(s^2-1)(t^4-2)\,dr\,ds\,dt$$

$$= \int_{-1}^{1}\int_{-1}^{1} (s^2-1)(t^4-2)\left(\frac{r^3}{3}\right|_{-1}^{1} ds\,dt$$

$$= \frac{2}{3}\int_{-1}^{1}\int_{-1}^{1} (s^2-1)(t^4-2)\,ds\,dt:$$

$$= \frac{2}{3}\int_{-1}^{1} (t^4-2)\left(\frac{s^3}{3} - s\right|_{-1}^{1} dt$$

$$= -\frac{8}{9}\int_{-1}^{1} (t^4-2)\,dt = -\frac{8}{9}\left(\frac{t^5}{5} - 2t\right|_{-1}^{1}$$

$$= \frac{16}{5} = 3.2$$

<u>6.34</u> THE GAUSS INTEGRATION FOR EACH OF THESE PROBLEMS WAS ACCOMPLISHED IN SOFTWARE USING A "LOOK-UP" TABLE FOR THE INTEGRATION POINTS AND WEIGHTING FACTORS. THE PROGRAM IS NOT INCLUDED HERE SINCE SO MANY DIFFERENT PACKAGES CAN BE USED (MATLAB, MATHCAD, EXCEL, ETC, ETC.) IN EACH PROBLEM, THE RESULT IS EXACT.

(a) $\int_0^3 (x^2-1)\,dx$ LET $r = \frac{2}{3}x - 1$

THEN $x = \frac{3(r+1)}{2}$ And $dx = \frac{3}{2}dr$

SO THE INTEGRAL BECOMES

$$\frac{3}{2}\int_{-1}^{1} \left[\left(\frac{3(r+1)}{2}\right)^2 - 1 \right] dr$$

TWO POINTS ARE REQUIRED:

$$r_{1/2} = \pm 0.5773503\ldots \qquad W_{1/2} = 1$$

AND THE RESULT IS

$$\int_0^3 (x^2-1)\,dx = 6 \qquad (5.9999\ldots)$$

(b) $\displaystyle\int_1^6 (y^3+2y)\,dy$

LET $r = \frac{1}{5}(2y-7)$ $\quad dy = \frac{5}{2}dr \quad y = \frac{1}{2}(5r+7)$

SO THE INTEGRAL IS THE SAME AS

$$\frac{5}{2}\int_{-1}^{1}\left[\frac{1}{8}(5r+7)^3 + (5r+7)\right]dr$$

AS IN (a), $r_{1/2} = \pm 0.57735\ldots$ $\quad W_{1/2} = 1$ SO

$$\int_1^6 (y^3+2y)\,dy = 358.75$$

(c) $\displaystyle\int_{-1}^{1} (4r^3+r)\,dr$ REQUIRES 2 POINTS AS

IN (a) AND (b) AND EVALUATES TO

$$\int_{-1}^{1} (4r^3+r)\,dr = 0$$

(d) $\int_{-1}^{1} (r^4+3r^2)\,dr$ REQUIRES 3 INTEGRATION POINTS FOR AN EXACT VALUE.

$r_1 = 0 \qquad W_1 = 0.8888\ldots$

$r_{2/3} = \pm\, 0.77459669\ldots \qquad W_{2/3} = 0.55555\ldots$

$$\int_{-1}^{1} (r^4+3r^2)\,dr = 2.4$$

(e) $\int_{-1}^{1} (r^4+r^3+r^2+r+1)\,dr$

REQUIRES 3 INTEGRATION POINTS AS IN (d) AND YIELDS THE VALUE 3.06666...

6.35 NOTE AT THE BEGINNING OF PROB. 6.34 SOLUTION IS APPLICABLE HERE ALSO.

(a) $\int_{0}^{1}\int_{0}^{2} xy\,dx\,dy$ LET $r = x-1$, $s = 2y-1$

THEN $dx = dr \qquad dy = \frac{ds}{2}$

AND THE INTEGRAL BECOMES

$$\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}(r+1)(s+1)\,dr\,ds$$

REQUIRING ONE INTEGRATION POINT FOR EACH VARIABLE WITH

$$r_1 = 0 \quad W_{r_1} = 2$$

$$S_1 = 0 \quad W_1 = 2$$

YIELDING

$$\int_0^1\int_0^2 xy\,dx\,dy = 1$$

(b) $$\int_{-1}^{1}\int_{-1}^{1}(r^2+2rs+s^2)\,dr\,ds$$

THE INTEGRAND IS QUADRATIC IN r AND S SO TWO INTEGRATION POINTS FOR EACH VARIABLE ARE NEEDED

$$r_{1/2} = s_{1/2} = \pm 0.57735 \ldots \quad W_{1/2} = 1$$

$$\int_{-1}^{1}\int_{-1}^{1}(r^2+2rs+s^2)\,dr\,ds = 2.6666\ldots$$

(c) $\int_{-1}^{1}\int_{-1}^{1} (r^3-1)(s^2+s)\,dr\,ds$

REQUIRES TWO INTEGRATION POINTS FOR EACH VARIABLE.

$r_{1/2} = s_{1/2} = \pm 0.57735\ldots \quad W_{1/2} = 1$

$$\int_{-1}^{1}\int_{-1}^{1} (r^3-1)(s^2+s)\,dr\,ds = -1.333\ldots$$

(d) $\int_{-1}^{1}\int_{-1}^{1} (r^5-2r^3)(s^3+s)\,dr\,ds$

REQUIRES THREE INTEGRATON POINTS FOR r AND TWO INTEGRATION POINTS FOR S. THE RESULT IS IDENTICALLY ZERO. (THE ASTUTE STUDENT WILL OBSERVE THIS IMMEDIATELY SINCE THE INTEGRAND IS AN ODD FUNCTION AND THE INTEGRATION LIMITS ARE SYMMETRIC.)

(e) $\int_{-1}^{1}\int_{-1}^{1} (6r^4-1)(s^2+s+1)\,dr\,ds$

3 INTEGRATION POINTS FOR r, 2 INTEGRATION POINTS FOR S $\Rightarrow$ 6 EVALUATIONS

$$\int_{-1}^{1}\int_{-1}^{1}(6r^4-1)(s^2+s+1)\,dr\,ds = 1.0666\ldots$$

6.36

(a) $\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz \qquad r=x-1 \quad s=y-1 \quad t=z-1$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(r+1)(s+1)(t+1)\,dr\,ds\,dt$$

ONE INTEGRATION POINT PER VARIABLE

$$r_1 = s_1 = t_1 = 0 \qquad W_{r_1} = W_{s_1} = W_{t_1} = 2$$

$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = 6$$

(b) $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} t(r-1)(s^2-2)\,dr\,ds\,dt$

ONE POINT INTEGRATION ON r AND t, TWO POINT INTEGRATION ON S

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} t(r-1)(s^2-2)\,dr\,ds\,dt = 0$$

(c) $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} t^3 r^3 s^2 \, dr\, ds\, dt$

TWO POINT INTEGRATION FOR EACH VARIABLE, THUS SIX EVALUATIONS

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} t^3 r^3 s^2 \, dr\, ds\, dt = 0$$

(d) $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} t^2 (r-2)^4 (s^2-1) \, dr\, ds\, dt$

3 POINT INTEGRATION FOR r

2 POINT INTEGRATION FOR s AND t

THUS 12 EVALUATIONS ARE REQUIRED

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} t^2 (r-2)^4 (s^2-1) \, dr\, ds\, dt = 43.3777\ldots$$

(e) $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (r^3-r)(s+4)(t^2-1) \, dr\, ds\, dt = 0$

6.37 TWO GAUSS POINTS IN EACH CASE

$r_{1/2} = \pm 0.57735\ldots \qquad W_{1/2} = 1$

(a) $\int_{-1}^{1} \cos^2 \pi r \, dr = 0.11579$

EXACT:

$$\int_{-1}^{1} \cos^2 \pi r \, dr = \left[\frac{1}{2\pi} \sin \pi r \cos \pi r + \frac{1}{2} r \right|_{-1}^{1} = 1$$

$\therefore$ THE TWO-POINT INTEGRATION IS A POOR ESTIMATE.

(b) $\int_{-1}^{1} \frac{r}{r^2+1} dr = 0$

AND THIS IS EXACT SINCE THE INTEGRAND IS ODD.

(c) $\int_{-1}^{1} \sin \pi r \cos \pi r \, dr = 0$ (EXACT)

(d) $\int_{-1}^{1}\int_{-1}^{1} \frac{r^2 s}{(r^3+s^2)} dr\, ds = 0$

EXACT: LET $u = r^3 + s^2$, THEN $du = 3r^2$

$r = -1 \qquad u = s^2 - 1$

$r = 1 \qquad u = s^2 + 1$

THE INTEGRAL BECOMES

$$\int_{-1}^{1}\int_{s^2-1}^{s^2+1} \frac{s}{u}\, du\, ds$$

$$= \int_{-1}^{1} s \ln(s^2+1)\, ds - \int_{-1} s \ln(s^2-1)\, ds$$

$$= \frac{1}{2}\int_{2}^{2} \ln v\, dv - \frac{1}{2}\int_{0}^{0} \ln v\, dv = 0$$

$\therefore$ RESULT IS EXACT

<u>6.38</u> THREE GAUSS POINTS:

$r_1 = 0 \qquad W_1 = 0.888\ldots$

$r_{2/3} = \pm 0.77459666\ldots \qquad W_{2/3} = 0.555\ldots$

(a) $\int_{-1}^{1} \cos^2 \pi r\, dr$ BECOMES 1.52996

THE NUMERICAL PROCEDURE FOR THIS INTEGRAND IS OSCILLATORY $\Rightarrow$ A GOOD REASON FOR NOT USING TRIGONOMETRIC FUNCTIONS IN INTERPOLATION FUNCTIONS.

(b) AND (c) EVALUATE TO ZERO AS IN PROBLEM 6.37.

(d) IS AN INTERESTING CASE. IN THE 3-POINT INTEGRATION, THE INTEGRAND MUST BE EVALUATED AT $r=0$, $s=0$ AND BECOMES $\frac{0}{0}$ $\Rightarrow$ UNDEFINED. THE GAUSS INTEGRATION PROCEDURE FAILS IN THIS CASE SINCE THE INTEGRAND IS NOT CONTINUOUS IN THE INTERVAL $(-1,1)$.

6.39

$$\int_{-1}^{1} \left(r^3 + 2r^2 + 1\right) dr$$

$$= \left(\frac{r^4}{4} + \frac{2r^3}{3} + r\right|_{-1}^{1} = \frac{10}{3}$$

TWO-POINT RESULT IS EXACT.

THREE-POINT RESULT IS $10/3$ $\Rightarrow$ ALSO EXACT.

$\therefore$ USING "TOO MANY" POINTS IS OK.

CHAPTER 9

9.1

$$\sigma_z = 0 \quad \Rightarrow \quad \epsilon_z = -\frac{\nu}{1-\nu}\left(\epsilon_x + \epsilon_y\right)$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\epsilon_x + \nu\epsilon_y - \frac{\nu^2}{1-\nu}\left(\epsilon_x + \epsilon_y\right)\right]$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[\left(1-\nu-\frac{\nu^2}{1-\nu}\right)\epsilon_x + \left(\nu - \frac{\nu^2}{1-\nu}\right)\epsilon_y\right]$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[\frac{(1-\nu)^2-\nu^2}{1-\nu}\,\epsilon_x + \frac{\nu(1-\nu)-\nu^2}{1-\nu}\,\epsilon_y\right]$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[\frac{1-2\nu}{1-\nu}\,\epsilon_x + \frac{\nu(1-2\nu)}{1-\nu}\,\epsilon_y\right]$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right)$$

σ_y DERIVATION IS SIMILAR

9.2

LET $C = \{z\}^T[A]\{z\}$

$$\{B\} = [A]\{z\}$$

$$B_i = \sum_{j=1}^{N} A_{ij} z_j$$

$$C = \{z\}^T\{B\} = \sum_{K=1}^{N} z_K B_K$$

$$C = \sum_{K=1}^{N} z_K \left(\sum_{j=1}^{N} A_{Kj} z_j \right)$$

$$C = \sum_{K=1}^{N} \sum_{j=1}^{N} A_{Kj} z_j z_K$$

9.3 THE RESULT IS TRIVIAL WITH $\epsilon_z = 0$. IT IS IMPORTANT TO NOTE THAT

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} (\epsilon_x + \epsilon_y)$$

IS <u>NOT</u> ZERO.

9.4 WRITE THE INTERPOLATION FUNCTIONS, EQ. 6.37 AS

$$N_1 = \alpha_1 + \beta_1 x + \gamma_1 y$$

$$N_2 = \alpha_2 + \beta_2 X + \gamma_2 Y$$

$$N_3 = \alpha_3 + \beta_3 X + \gamma_3 Y$$

$$\frac{\partial N_i}{\partial X} = \beta_i \qquad \frac{\partial N_i}{\partial Y} = \gamma_i$$

$$[B] = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

$[B]$ IS A CONSTANT MATRIX (SAME AS FOR PLANE STRESS)

9.5

RELATIONS ARE IDENTICAL TO THOSE GIVEN BY EQ. 9.63 THE POINT FOR THE STUDENT IS THAT THE STRAIN-DISPLACEMENT RELATIONS ARE NOT DEPENDENT ON THE STATE OF STRESS.

9.6 $N_i = \alpha_i + \beta_i x + \gamma_i y$

PER EQ. 9.16

$$U = \frac{V^{(e)}}{2} \{\delta\}^T [B]^T [D] [B] \{\delta\}$$

$$= \frac{1}{2} \{\delta\}^T [K^{(e)}] \{\delta\}$$

$[K^{(e)}]$ IS GIVEN BY EQ. 9.32

NOTING THAT $[K^{(e)}]$ IS FULLY POPULATED

$$U = \frac{1}{2}\Big[K_{11} u_1^2 + 2K_{12} u_1 u_2 + 2K_{13} u_1 u_3$$

$$+ 2K_{14} u_1 v_1 + 2K_{15} u_1 v_2 + 2K_{16} u_1 v_3 + K_{22} u_2^2$$

$$+ 2K_{23} u_2 u_3 + 2K_{24} u_2 v_1 + 2K_{25} u_2 v_2 + 2K_{26} u_2 v_3$$

$$+ K_{33} u_3^2 + 2K_{34} u_3 v_1 + 2K_{35} u_3 v_2 + 2K_{36} u_3 v_3$$

$$+ K_{44} v_1^2 + 2K_{45} v_1 v_2 + 2K_{46} v_1 v_3 + K_{55} v_2^2$$

$$+ 2K_{56} v_2 v_3 + K_{66} v_3^2 \Big]$$

9.7 $(n_x, n_y) \cong (-0.97, 0.243) \qquad p = -p_o$

ALONG 1-3: $N_1 = 1 - \frac{2}{3}x - \frac{1}{3}y$

$N_2 = 0$

$N_3 = \frac{2}{3}y - \frac{2}{3}x$

$y = 4x \qquad dS = \dfrac{dx}{0.243} \qquad N_1 = 1 - 2x \quad N_3 = 2x$

$$f_{1x} = 0.97\, p_o t \int_0^{0.5} (1-2x)\frac{dx}{0.243} \cong p_o t$$

$$f_{1y} = -0.243\, p_o t \int_0^{0.5} (1-2x)\frac{dx}{0.243} \cong -0.25\, p_o t$$

$$f_{3x} = 0.97\, p_o t \int_0^{0.5} 2x\, \frac{dx}{0.243} \cong p_o t$$

$$f_{3y} = -0.243\, p_o t \int_0^{0.5} 2x\, \frac{dx}{0.243} \cong -0.25\, p_o t$$

9.8 $(n_x, n_y) \cong (-0.992, -0.124)$

PATH LENGTH FROM NODE 1 TO NODE 3 = $\sqrt{0.65}$

$$p = -p_0\left(1 - \frac{s}{\sqrt{0.65}}\right)$$

ALONG 1-3 $y = -8x$ AND $N_1 = 1 + 10x$

$N_2 = 0$

$N_3 = -10x$

$$s = -\frac{x}{0.124} \qquad ds = -\frac{dx}{0.124}$$

$$f_{1x} = 0.992\, p_0 t \int \left(1 - \frac{s}{\sqrt{0.65}}\right) N_1\, ds$$

$$= -0.992\, p_0 t \int_0^{-0.1} \left(1 + \frac{x}{0.124\sqrt{0.65}}\right)(1 + 10x)\frac{dx}{0.124}$$

$$= -0.992\, p_0 t \int_0^{-0.1} (1 + 10x)^2 \frac{dx}{0.124}$$

$$f_{1x} \cong 0.267\, p_0 t$$

SIMILARLY

$$f_{1y} = 0.124\, p_o t \int \left(1 - \frac{s}{\sqrt{0.65}}\right) N_1\, ds$$

$$f_{1y} \cong 0.033\, p_o t$$

$$f_{3x} = 0.992\, p_o t \int \left(1 - \frac{s}{\sqrt{0.65}}\right) N_3\, ds$$

$$= 0.992\, p_o t \int_0^{-0.1} (1+10x)(10x)\, \frac{dx}{0.124}$$

$$f_{3x} \cong 0.133\, p_o t$$

SIMILARLY f_{3y} IS FOUND TO BE

$$f_{3y} \cong 0.017\, p_o t$$

9.9 THE INTERPOLATION FUNCTIONS ARE

$$N_1 = 1 - \frac{2}{3}x - \frac{1}{3}y$$

$$N_2 = \frac{4}{3}x - \frac{1}{3}y$$

$$N_3 = \frac{2}{3}y - \frac{2}{3}x$$

HENCE

$$[B] = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \end{bmatrix}$$

$$[D] = \frac{15(10^6)}{1-0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$[K^{(e)}] = V^{(e)}[B]^T[D][B]$$

$$= A^{(e)} t\, [B]^T[D][B]$$

$$A^{(e)} = \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0.5 & 2 \end{vmatrix} = 0.75$$

VIA A RELATIVELY SIMPLE MATLAB PROGRAM

$$[K^{(e)}] = 12.36(10^6)\begin{bmatrix} 0.1500 & -0.4056 & 0.1444 & 0.1111 & -0.1222 & 0.0111 \\ -0.4056 & 1.8167 & -0.9667 & -0.0556 & -0.2889 & 0.3444 \\ 0.1444 & -0.9667 & 0.6000 & -0.0889 & 0.3778 & -0.2889 \\ 0.1111 & -0.0556 & -0.0889 & 0.2667 & -0.2000 & -0.0667 \\ -0.1222 & -0.2889 & 0.3778 & -0.2000 & 0.7333 & -0.5333 \\ 0.0111 & 0.3444 & -0.2889 & -0.0667 & -0.5333 & 0.6000 \end{bmatrix}$$

FOR $t = 1$.

9.10 FOR PLANE STRAIN

$$[D] = \frac{15(10^6)}{1.3(0.4)}\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$[K^{(e)}] = 21.63(10^6)\begin{bmatrix} 0.1000 & -0.2889 & 0.1111 & 0.0778 & -0.0556 & -0.0222 \\ -0.2889 & 1.2667 & -0.6667 & -0.0889 & -0.2222 & 0.3111 \\ 0.1111 & -0.6667 & 0.4000 & -0.0222 & 0.2444 & -0.2222 \\ 0.0778 & -0.0889 & -0.0222 & 0.1667 & -0.1000 & -0.0667 \\ -0.0556 & -0.2222 & 0.2444 & -0.1000 & 0.4333 & -0.3333 \\ -0.0222 & 0.3111 & -0.2222 & -0.0667 & -0.3333 & 0.4000 \end{bmatrix}$$

9.11 FOR THE AXISYMMETRIC CASE

$$N_1 = 1 - \frac{2}{3}r - \frac{1}{3}z$$

$$N_2 = \frac{4}{3}r - \frac{1}{3}z$$

$$N_3 = \frac{2}{3}z - \frac{2}{3}r$$

AND PER EQ. 9.97

$$[B] = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{r} - \frac{2}{3} - \frac{z}{3r} & \frac{4}{3} - \frac{z}{3r} & \frac{2}{3}\left(\frac{z}{r} - 1\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \end{bmatrix}$$

OWING TO THE TANGENTIAL STRAIN ϵ_θ, INTEGRATION OF SOME TERMS IS REQUIRED SINCE

$$[K^{(e)}] = 2\pi \iint_{A^{(e)}} [B]^T [D] [B] \, r \, dr \, dz$$

THIS PARTICULAR PROBLEM IS NOT TRACTABLE AS $r = 0$ IS A NODAL COORDINATE. THE AUTHOR APOLOGIZES FOR THIS OVERSIGHT.

<u>9.12 & 9.13</u>

$$\{\delta\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0.003 \\ 0.001 \\ 0.0015 \\ 0 \\ 0.0005 \\ 0 \end{Bmatrix}$$

$\{\epsilon\} = [B]\{\delta\}$ WHERE $[B]$ IS KNOWN FROM PROBLEM 9.9 GIVES

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} -0.667 \\ -0.1667 \\ 0.333 \end{Bmatrix} (10^{-3})$$

$$\{\sigma\} = [D]\{\epsilon\}$$

PLANE STRESS: $(t = 1)$

$$[D] = \frac{15(10^6)}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$\{\sigma\} = [D]\{\epsilon\} = \begin{Bmatrix} -8.86 \\ -4.53 \\ 1.44 \end{Bmatrix}(10^3)$$

$$U = \frac{1}{2}\{\delta^T\}[K]\{\delta\} \cong 3.57$$

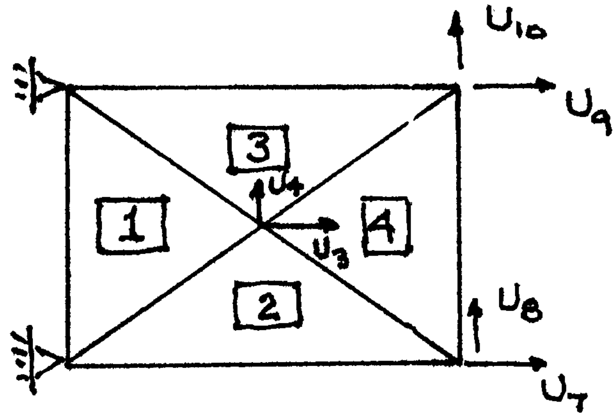
PLANE STRAIN:

$$[D] = \frac{15(10^6)}{1.3(0.4)}\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$\{\sigma\} = [D]\{\epsilon\} = \begin{Bmatrix} -14.90 \\ -9.14 \\ 3.85 \end{Bmatrix}(10^3)$$

$$U = \frac{1}{2}\{\delta\}^T[K]\{\delta\} \cong 6.05$$

9.14



VIA THE STUDENT EDITION OF ANSYS, THE DISPLACEMENTS ARE

$$U_3 = -0.317(10^{-3})$$
$$U_4 = -0.536(10^{-3})$$
$$U_7 = -0.402(10^{-3})$$
$$U_8 = -0.117(10^{-2}) \quad \text{IN.}$$
$$U_9 = 0.462(10^{-3})$$
$$U_{10} = -0.130(10^{-2})$$

THE ELEMENT STRESSES ARE: (IN PSI)

ELEMENT 1

$\sigma_x = -394.62 \quad \sigma_y = 24.951 \quad \tau_{xy} = -220.35$

ELEMENT 2

$\sigma_x = -6.97 \quad \sigma_y = -2.09 \quad \tau_{xy} = -411.92$

ELEMENT 3

$\sigma_x = 6.97 \quad \sigma_y = -197.24 \quad \tau_{xy} = -38.1$

ELEMENT 4

$\sigma_x = 394.62 \quad \sigma_y = -224.28 \quad \tau_{xy} = -229.65$

THE WORK OF THE NODAL LOADING IS

$$W = \tfrac{1}{2}(4500)(0.130)(10^{-2}) = 2.93 \text{ IN}\cdot\text{LB}$$

AND THE TOTAL STRAIN ENERGY IS

$$U_e = \frac{1}{2}\sum_{i=1}^{4}\left(\frac{\sigma_x^2}{E} + \frac{\sigma_y^2}{E} + \frac{\tau_{xy}^2}{G}\right)V$$

WHERE V IS ELEMENT VOLUME AND IN THIS CASE IS 150 IN^3 FOR EACH ELEMENT. SUBSTITUTING THE STRESS VALUES GIVES

$$U_e \cong 2.774 \text{ IN}\cdot\text{LB}$$

9.15

$$K_{11} = \frac{Etb}{16a(1+2r)} \int_{-1}^{1}\int_{-1}^{1} (s-1)^2\, dr\, ds$$

$$+ \frac{Eta}{32b(1+r)} \int_{-1}^{1}\int_{-1}^{1} (r-1)^2\, dr\, ds$$

$$\int_{-1}^{1}\int_{-1}^{1} (s-1)^2\, dr\, ds = \int_{-1}^{1}\int_{-1}^{1} (r-1)^2\, dr\, ds$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j (s_i - 1)^2$$

$$W_1 = W_2 = 1.0 \qquad s_i = r_j = \pm \frac{\sqrt{3}}{3}$$

$$= 2\left[\left(\frac{\sqrt{3}}{3} - 1\right)^2 + \left(\frac{\sqrt{3}}{3} + 1\right)^2\right] = \frac{16}{3}$$

9.16

$$N_1 = \frac{7}{3} - \frac{2}{15} r - \frac{1}{30} z$$

$$N_2 = \frac{1}{15} r - \frac{1}{30} z - \frac{2}{3}$$

$$N_3 = \frac{1}{15} r + \frac{1}{15} z - \frac{2}{3}$$

$$[B] = \begin{bmatrix} -\frac{2}{15} & \frac{1}{15} & \frac{1}{15} & 0 & 0 & 0 \\ \frac{N_1}{r} & \frac{N_2}{r} & \frac{N_3}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{30} & -\frac{1}{30} & \frac{1}{15} \\ -\frac{1}{30} & -\frac{1}{30} & \frac{1}{15} & -\frac{2}{15} & \frac{1}{15} & \frac{1}{15} \end{bmatrix}$$

$$[D] = 1.578(10)^3 \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$[K^{(e)}] = 2\pi \iint_{A^{(e)}} [B]^T [D] [B] \, r \, dr \, dz$$

AT THIS POINT, THE STUDENT WILL REALIZE THE MATHEMATICAL COMPLEXITY OF COMPUTING EXACT VALUES VIA DIRECT INTEGRATION. AND THAT IS THE INTENT OF THIS PROBLEM, THE AUTHOR DOES NOT EXPECT THAT THE STUDENT WILL PERFORM ALL OF THE TEDIOUS INTEGRATIONS REQUIRED!

CENTROIDAL APPOXIMATION:

$$[K^{(e)}] \cong 2\pi \bar{r} A [\bar{B}]^T [D] [\bar{B}]$$

WHERE $[\bar{B}]$ IS EVALUATED AT $(\bar{r}, \bar{z})$

$$A = \frac{1}{2} \begin{vmatrix} 1 & 10 & 0 \\ 1 & 20 & -15 \\ 1 & 15 & 10 \end{vmatrix} = 75 \text{ MM}^2$$

$$\bar{r} = \frac{r_1 + r_2 + r_3}{3} = 15 \text{ MM}$$

$$\bar{z} = \frac{z_1 + z_2 + z_3}{3} = 0$$

$$\frac{N_1(\bar{r}, \bar{z})}{\bar{r}} = \frac{N_2(\bar{r}, \bar{z})}{\bar{r}} = \frac{N_3(\bar{r}, \bar{z})}{\bar{r}} = \frac{1}{3}$$

$$[\bar{B}] = \begin{bmatrix} -\frac{2}{15} & \frac{1}{15} & \frac{1}{15} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{30} & -\frac{1}{30} & \frac{1}{15} \\ -\frac{1}{30} & -\frac{1}{30} & \frac{1}{15} & -\frac{2}{15} & \frac{1}{15} & \frac{1}{15} \end{bmatrix}$$

$$[K^{(e)}] \cong 2\pi(15)(75)\,[\bar{B}]^T[D][B]$$

$$[K^{(e)}] = \frac{10^6}{9}\begin{bmatrix} 6.402396 & 6.536244 & 6.46932 & -0.11154 & -0.245388 & 0.356928 \\ 6.536244 & 9.4809 & 9.413976 & -0.312312 & -0.44616 & 0.758472 \\ 6.46932 & 9.413976 & 9.547824 & -0.580008 & -0.312312 & 0.89232 \\ -0.11154 & -0.312312 & -0.580008 & 0.435006 & -0.100386 & -0.33462 \\ -0.245388 & -0.44616 & -0.312312 & -0.100386 & 0.16731 & -0.066924 \\ 0.356928 & 0.758472 & 0.89232 & -0.33462 & -0.066924 & 0.401544 \end{bmatrix} \text{LB/IN}$$

OF COURSE, AS ELEMENT SIZE DECREASES, THE CENTROIDAL APPROXIMATION BECOMES MORE ACCURATE. WHILE I HAVE NOT COVERED THE TOPIC IN THE TEXT, THE INSTRUCTOR MAY WISH TO DISCUSS NUMERICAL INTEGRATION IN TRIANGULAR REGIONS.

9.17 FOR FACE 1-3:

$(n_r, n_z) = (-0.894, 0.447)$

$z = 2(r-10) \quad N_1 = 3 - \frac{1}{5}r \quad N_2 = 0 \quad N_3 = \frac{1}{5}r - 2$

$$dS = \frac{dr}{0.447} = \frac{dz}{0.894}$$

$$f_{1r} = 0.894\, p_o\, (2\pi) \int_{10}^{15} \left(3 - \frac{1}{5}r\right) r \,\frac{dr}{0.447}$$

$$f_{1r} = 366.5\, p_o$$

$$f_{1z} = -0.447\, p_o\, (2\pi) \int_{10}^{15} \left(3 - \frac{1}{5}r\right) r \,\frac{dr}{0.447}$$

$$f_{1z} = -183.2\, p_o$$

$$f_{3r} = 0.894\, p_o\, (2\pi) \int_{10}^{15} \left(\frac{1}{5}r - 2\right) r \,\frac{dr}{0.447}$$

$$f_{3r} = 418.9\, p_o$$

$$f_{3z} = 0.894\, p_0 (2\pi) \int_{10}^{15} \left(\tfrac{1}{5} r - 2\right) r \frac{dr}{0.894}$$

$$f_{3z} = 209.4\, p_0$$

(ABOVE SOLUTION ASSUMES p_0 IS GIVEN IN MPa.)

9.18

$$\omega = \frac{2\pi(3600)}{60} \cong 377 \text{ RAD/S}$$

$$R_B = (377)^2 r \quad \text{MM/S}^2$$

$$f_{ri} = 2\pi\rho \iint_A N_i R_B\, r\, dr\, dz \qquad i = 1,2,3$$

$$= 2\pi\rho (377)^2 \iint_A N_i\, r^2\, dr\, dz \qquad i = 1,2,3$$

USING AREA COORDINATES

$$f_{ri} = 2\pi\rho (377)^2 \iint_A L_i (r_1 L_1 + r_2 L_2 + r_3 L_3)\, dA$$

$$f_{ri} = 2\pi\rho\,(377)^2 \iint_A L_i \left(r_1^2 L_1^2 + 2 r_1 r_2 L_1 L_2 \right.$$

$$\left. + 2 r_1 r_3 L_1 L_3 + r_2^2 L_2^2 + 2 r_2 r_3 L_2 L_3 + r_3^2 L_3^2 \right) dA$$

<u>$i = 1$</u>

$$f_{r1} = 2\pi\rho\,(377)^2 (2)(75) \left[\frac{r_1^2}{20} + \frac{2 r_1 r_2}{60} + \frac{2 r_1 r_3}{60} + \frac{r_2^2}{60} \right.$$

$$\left. + \frac{2 r_2 r_3}{120} + \frac{r_3^2}{60} \right]$$

$f_{r1} \cong 4.30\,(10^8)\,\rho$ N. $\quad$ (ρ IN $^{kg}/_{mm^3}$)

OR $0.430\,\rho$ N. IF ρ ~~IN~~ $^{kg}/_{m^3}$

<u>$i = 2$</u>

$$f_{r2} = 2\pi\rho\,(377)^2 (2)(75) \left[\frac{r_1^2}{60} + \frac{2 r_1 r_2}{60} + \frac{2 r_1 r_3}{120} \right.$$

$$\left. + \frac{r_2^2}{20} + \frac{2 r_2 r_3}{60} + \frac{r_3^2}{60} \right]$$

$f_{r2} \cong 5.97\,(10^8)\,\rho$ N. $\quad$ (ρ IN $^{kg}/_{mm^3}$)

OR $0.597\,\rho$ N $\quad$ (ρ IN $^{kg}/_{m^3}$)

$\underline{i = 3}$

$$f_{r3} = 2\pi\rho(377)^2(2)(75)\left[\frac{r_1^2}{60} + \frac{2r_1r_2}{120} + \frac{2r_1r_3}{60} + \frac{r_2^2}{60} + \frac{2r_2r_3}{60} + \frac{r_3^2}{20}\right]$$

$$f_{r3} = 5.08(10^8)\rho \text{ N.} \qquad (\rho \text{ in kg/mm}^3)$$

$$\text{OR } 0.508\rho \text{ N.} \qquad (\rho \text{ in kg/m}^3)$$

9.19

a) THE CUBIC POLYNOMIAL

$$a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5xy + a_6y^2 + a_7yz + a_8z^2 + a_9xz + a_{10}x^3 + a_{11}x^2y + a_{12}y^2x + a_{13}y^3 + a_{14}x^2z + a_{15}z^2x + a_{16}z^3 + a_{17}y^2z + a_{18}yz^2 + a_{19}xyz$$

IS APPROPRIATE FOR REPRESENTATION OF $u(x,y,z)$, $v(x,y,z)$ AND $w(x,y,z)$.

b) BASED ON THE POLYNOMIAL IN a), THE FIRST PARTIAL DERIVATIVES, THUS STRAIN COMPONENTS, VARY QUADRATICALLY.

c) THE STIFFNESS MATRIX IS 60×60.

<u>9.20</u> IN A UNIAXIAL TENSION TEST AT YIELDING

$$\sigma_1 = S_Y \quad \sigma_2 = \sigma_3 = 0$$

$$U_e = \frac{S_Y^2}{2E} V \qquad (9.123)$$

$$\sigma_{AV} = \frac{S_Y}{3}$$

$$U_{hyd} = \frac{S_Y^2}{6E}(1-2\nu)V$$

$$U_d = \frac{S_Y^2}{2E} V - \frac{S_Y^2}{6E}(1-2\nu)V$$

$$U_d = \frac{1+\nu}{3E} S_Y^2 V$$

9.21

$$\sigma_e = \left[\frac{(200-0)^2 + (200-(-90))^2 + (0-(-90))^2}{2} \right]^{1/2}$$

$$\sigma_e \cong 257.1 \text{ MPa}$$

YIELDING IS NOT INDICATED.

$$FoS = \frac{270}{257.1} \cong 1.05$$

9.22

$$\tau_{MAX} = \frac{|\sigma_1 - \sigma_3|}{2} = 145 \text{ MPa}$$

$$S_{ys} = \frac{S_y}{2} = 135 \text{ MPa}$$

YIELDING IS INDICATED BY MSST.

9.23

$$\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = -2G\theta$$

$$\tau_{xz} = -\frac{\partial \varphi}{\partial y} \qquad \tau_{xy} = \frac{\partial \varphi}{\partial z}$$

a) THE SHEAR STRESS COMPONENTS ARE ANALOGOUS TO THE HEAT FLUX

COMPONENTS q_x AND q_y WHEN THE THERMAL CONDUCTIVITIES HAVE VALUE OF UNITY.

b) $$T = 2 \iint_A \psi \, dA = \text{TORQUE}$$

IN THE THERMAL ANALOGY, ψ CORRESPONDS TO TEMPERATURE SO

$$\text{TORQUE} = 2 \sum_{e=1}^{n} \left(\iint_{A^{(e)}} [N^{(e)}] \, dA^{(e)} \{T^{(e)}\} \right)$$

WHERE n = TOTAL NUMBER OF ELEMENTS

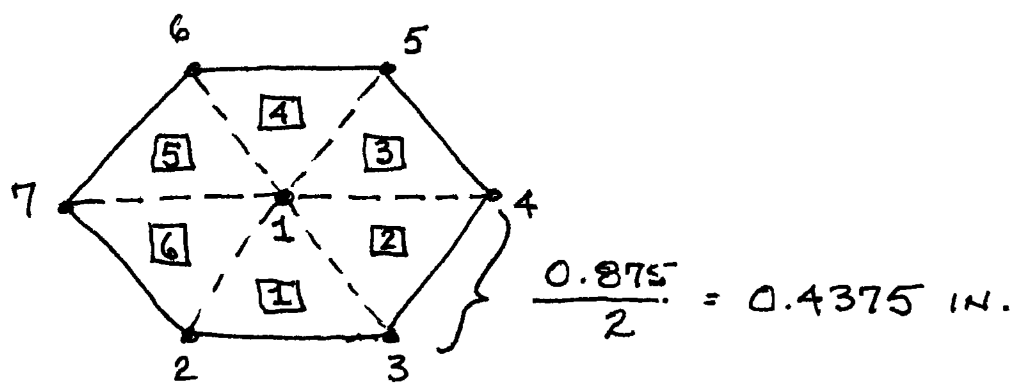
$\{T^{(e)}\}$ = ELEMENT NODAL TEMPERATURES

9.24 THE TORSION PROBLEM CAN BE CONSIDERED AS A GENERAL STATE OF STRESS AND ANY 3D ELASTIC SOLID ELEMENT CAN BE USED TO MODEL THE PROBLEM. NODAL FORCES IN A PLANE OR PLANES PERPENDICULAR TO THE AXIS OF TORSION CAN BE USED TO REPRESENT

THE APPLIED TORQUE PROVIDED THAT THE NODAL FORCES FORM COUPLES TO PRECLUDE BENDING AND PRODUCE ZERO NET APPLIED FORCE.

9.25 $G = 12(10^6)$ psi $L = 12''$

$T = 250$ FT·LB $\Rightarrow 3000$ IN·LB.



USING SIX (EQUILATERAL) TRIANGULAR ELEMENTS AS SHOWN, WE NEED ONLY COMPUTE ψ_1 SINCE

$$\psi_2 = \psi_3 = \psi_4 = \psi_5 = \psi_6 = \psi_7 = 0$$

PER EQ. 9.145. TAKING THE ORIGIN AT NODE 1 WITH X-AXIS THROUGH NODE 4, THE INTERPOLATION FUNCTIONS FOR ELEMENT 1 ARE:

$$N_1 = 1 + \frac{16}{7}Y \qquad N_2 = -\frac{16}{7}x - \frac{8}{7}Y$$

$$N_3 = \frac{16}{7}x - \frac{8}{7}Y$$

$$\left[\frac{\partial N}{\partial x}\right]^T = \begin{bmatrix} 0 \\ -\frac{16}{7} \\ \frac{16}{7} \end{bmatrix}$$

$$\left[\frac{\partial N}{\partial x}\right]^T \left[\frac{\partial N}{\partial x}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{256}{49} & -\frac{256}{49} \\ 0 & -\frac{256}{49} & \frac{256}{49} \end{bmatrix}$$

$$\left[\frac{\partial N}{\partial y}\right]^T = \begin{bmatrix} 16/7 \\ -8/7 \\ -8/7 \end{bmatrix}$$

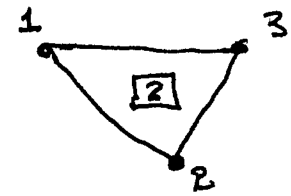
$$\left[\frac{\partial N}{\partial y}\right]^T \left[\frac{\partial N}{\partial y}\right] = \begin{bmatrix} \frac{256}{49} & -\frac{128}{49} & -\frac{128}{49} \\ -\frac{128}{49} & \frac{64}{49} & \frac{64}{49} \\ -\frac{128}{49} & \frac{64}{49} & \frac{64}{49} \end{bmatrix}$$

$$[K^{(e)}] = \iint_A \left(\left[\frac{\partial N}{\partial x}\right]^T \left[\frac{\partial N}{\partial x}\right] + \left[\frac{\partial N}{\partial y}\right]^T \left[\frac{\partial N}{\partial y}\right] \right) dA$$

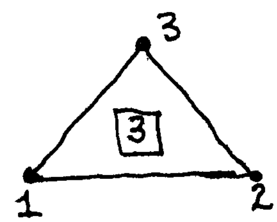
$$[K^{(1)}] = \begin{bmatrix} \frac{256}{49} & -\frac{128}{49} & -\frac{128}{49} \\ -\frac{128}{49} & \frac{320}{49} & -\frac{192}{49} \\ -\frac{128}{49} & -\frac{192}{49} & \frac{320}{49} \end{bmatrix} \left(\tfrac{1}{2}\right)(0.4375)^2$$

FOR THE OTHER ELEMENTS:

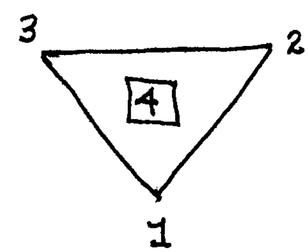
$$[K^{(2)}] = \frac{1}{49}\begin{bmatrix} 320 & -128 & -192 \\ -128 & 256 & -128 \\ -192 & -128 & 320 \end{bmatrix} \left(\tfrac{1}{2}\right)(0.4375)^2$$



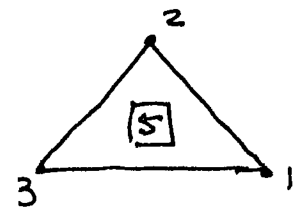
$$[K^{(3)}] = \frac{1}{49}\begin{bmatrix} 320 & -192 & -128 \\ -128 & 320 & -128 \\ -128 & -128 & 256 \end{bmatrix} \left(\tfrac{1}{2}\right)(0.4375)^2$$



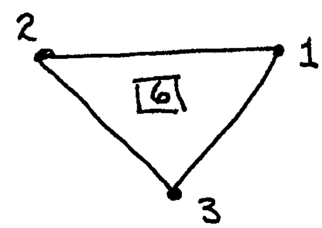
$$[K^{(4)}] = \frac{1}{49}\begin{bmatrix} 256 & -128 & -128 \\ -128 & 320 & -192 \\ -128 & -192 & 320 \end{bmatrix} \left(\tfrac{1}{2}\right)(0.4375)^2$$



$$[K^{(5)}] = \frac{1}{49}\begin{bmatrix} 320 & -128 & -192 \\ -128 & 256 & -128 \\ -192 & -128 & 320 \end{bmatrix}\left(\tfrac{1}{2}\right)(0.4375)^2$$



$$[K^{(6)}] = \frac{1}{49}\begin{bmatrix} 320 & -192 & -128 \\ -192 & 320 & -128 \\ -128 & -128 & 256 \end{bmatrix}\left(\tfrac{1}{2}\right)(0.4375)^2$$



THE ELEMENT NODAL FORCES ARE, FOR EACH ELEMENT,

$$\{f^{(e)}\} = \frac{2G\phi A^{(e)}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \frac{2(12)(10^6)\,\phi\left(\tfrac{1}{2}\right)(0.4375)^2}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= 7.66(10^5)\,\phi\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

SINCE Ψ_1 IS THE ONLY UNKNOWN NODAL VARIABLE WE HAVE

$$K_{11}\Psi_1 = F_1$$

OR

$$\frac{1}{49}\left(\frac{1}{2}\right)(0.4375)^2\left(256+320+320+256+320+320\right)\Psi_1$$

$$= 6(7.66)(10^5)\phi$$

$$\Psi_1 = 1.31(10^6)\phi$$

$$T = 6\left(\frac{2A^{(e)}}{3}\Psi_1\right) \cong 5.01(10^5)\phi$$

$$\therefore \frac{T}{\phi} = 5.01(10^5)$$

$$\phi = \frac{3000}{5.01(10^5)} \cong 6(10^{-3}) \text{ RAD/IN.}$$

$$\theta = 4\phi = 7.2(10^{-2}) \text{ RAD}$$